INNOVATIVE FINITE ELEMENT METHODS FOR PLATES*

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Abstract. Finite element methods for the Reissner–Mindlin plate theory are discussed. Methods in which both the tranverse displacement and the rotation are approximated by finite elements of low degree mostly suffer from locking. However a number of related methods have been devised recently which avoid locking effects. Although the finite element spaces for both the rotation and transverse displacement contain little more than piecewise linear functions, optimal order convergence holds uniformly in the thickness. The main ideas leading to such methods are reviewed and the relationships between various methods are clarified.

1. The Reissner–Mindlin plate equations. The Reissner–Mindlin plate equations describe the bending of a linearly elastic plate in terms of the transverse displacement, ω, of the middle plane and the rotation, $\boldsymbol{\phi}$, of the fibers normal to the middle plane. This model, as well as its generalization to shells, is frequently used for plates and shells of small to moderate thickness. Assuming that the material is homogeneous and isotropic with Young's modulus E and Poisson ratio ν, the governing differential equations, which are to hold on the two dimensional region Ω occupied by the middle plane of the plate, take the form

$$-\operatorname{\mathbf{div}} \mathsf{C}\, \mathcal{E}(\boldsymbol{\phi}) - \lambda t^{-2}(\operatorname{\mathbf{grad}} \omega - \boldsymbol{\phi}) = 0, \tag{1}$$

$$-\lambda t^{-2} \operatorname{div}(\operatorname{\mathbf{grad}} \omega - \boldsymbol{\phi}) = g, \tag{2}$$

where t is the plate thickness, gt^3 is the transverse load force density per unit area, $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, $\lambda = Ek/2(1+\nu)$, and the fourth order tensor C is defined by

$$\mathsf{C}\mathcal{T} = \frac{E}{12(1-\nu^2)}\left[(1-\nu)\mathcal{T} + \nu \operatorname{tr}(\mathcal{T})\mathcal{I}\right]$$

for any 2×2 matrix $\mathcal{T}$ ($\mathcal{I}$ denotes the 2×2 identity matrix). The load function has been normalized by a factor of t^3 so that the solution tends to a nonzero limit as t tends to zero. The shear correction factor k is a constant often taken to equal $5/6$.

Solutions to this system are minimizers of the energy functional

$$(\boldsymbol{\phi}, \omega) \mapsto J(\boldsymbol{\phi}, \omega) := \int_\Omega \left[\frac{1}{2}\mathsf{C}\,\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\phi}) + \frac{1}{2}\lambda t^{-2} |\operatorname{\mathbf{grad}} \omega - \boldsymbol{\phi}|^2 - g\omega\right] d\boldsymbol{x},$$

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defined for $(\boldsymbol{\phi}, \omega) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$. If we minimize J over the subspace $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$ consisting of vector and scalar fields vanishing on the boundary of Ω, for example, we obtain the system (1)–(2) as the Euler–Lagrange equations together with the Dirichlet boundary conditions

$$\boldsymbol{\phi} = 0, \qquad \omega = 0 \qquad \text{on } \partial\Omega,$$

which models a hard clamped, or welded, plate. If we minimize J over $\boldsymbol{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ we obtain the natural boundary condition

$$\mathsf{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega$$

as well as the essential boundary condition $\omega = 0$. These boundary conditions model a (soft) simply-supported plate. There are numerous other possibilities as well, cf. [3]. For simplicity we shall mostly consider the hard clamped case, commenting on other boundary conditions when appropriate.

By restricting the trial space over which we seek a minimum of the energy functional J to a finite-dimensional subspace of $\boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ which incorporates the essential boundary conditions, we obtain a Galerkin discretization of the Reissner–Mindlin system. Since the energy functional J is positive definite, the standard theory of Galerkin methods implies that the error in the resulting approximate solution will be no greater than a constant multiple of the error in the best approximation of the exact solution from the trial space. Consequently, if a sequence of trial spaces is selected for which the best approximation error converges to zero, then the error in the Galerkin approximation will converge to zero at the same rate. The trial spaces are typically selected to be finite element spaces with convergence sought as the mesh is refined. A natural choice of finite element space, for example, is the space of vector and scalar fields which are continuous and piecewise linear over some triangulation of Ω. Unfortunately, this method (and indeed most finite element methods based on such "natural" choices of elements) performs very poorly for t small. The computed displacements are found to be much smaller than the true values. This phenomenon, known as *locking* of the numerical solution, is by now well-understood. It arises from the fact that for small t a minimizer of J must nearly satisfy the Kirchhoff constraint $\mathbf{grad}\,\omega - \boldsymbol{\phi} = 0$. For the exact problem, in which J is minimized over $\boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ (or some subspace thereof incorporating the essential boundary conditions), this condition is indeed met, and in the limit as t tends to 0 the Kirchhoff hypothesis is satisfied exactly. But if ω_h and $\boldsymbol{\phi}_h$ are continuous piecewise linears, then the relation $\mathbf{grad}\,\omega_h = \boldsymbol{\phi}_h$ implies that $\boldsymbol{\phi}$ is globally constant and ω_h globally linear. Together with the boundary condition $\omega_h = 0$, it is even necessary that $\boldsymbol{\phi}_h$ and ω_h vanish identically. For small but non-vanishing t, the minimizing $\boldsymbol{\phi}_h$ and ω_h will be forced to be nearly zero, and consequently unable to approximate $\boldsymbol{\phi}$ and ω accurately. This is, of course, not at odds with the mathematical theory. It merely indicates that the constant relating the error in the Galerkin approximation to the error in best approximation depends on

the thickness in such a manner as to tend to infinity as the thickness tends to zero. This reflects the fact that the energy functional, with its coefficient of λt^{-2}, degenerates as t tends to zero. Mathematically speaking, for each fixed t, $\boldsymbol{\phi}_h$ and ω_h do indeed converge to $\boldsymbol{\phi}$ and ω at the approximation theoretic optimal rate ($O(h)$ in H^1 and $O(h^2)$ in L^2 if h is the mesh size), but the convergence is not uniform in t. In fact it is easy to see that there is no uniform convergence (at any rate) for this method.

Many researchers have sought to devise finite element methods for the Reissner–Mindlin plate which do not deviate too much from the simple linear-linear element, but which avoid such locking phenomena. This involves both varying the choice of elements and modifying the variational formulation on which the Galerkin method is based. From a theoretical point of view, one would like a method for which approximation holds at the optimal rate (dictated by the approximation properties of the elements) *uniformly* in the thickness t. That is, the constant relating the error in the Galerkin approximation to the error in best approximation is, ideally, independent of t. From a practical point of view, the elements should be as simple as possible and the method as close as possible to a standard Galerkin method such as described above. Many elements have been proposed and good results have been reported on various test problems for a number of them. Until recently, however, none has been shown to provide uniform optimal order approximation. In this paper we shall consider the problem of devising simple uniformly optimal order finite elements for the Reissner–Mindlin plate, and review a number of ideas which have enabled the construction of a few such elements.

A list of notations is given at the end of the paper.

2. Mixed variational formulations. The first step is to introduce the shear force vector

$$\boldsymbol{\zeta} = \lambda t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi})$$

as an independent unknown. Then the triple $(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta})$ can be characterized as the unique critical point of the functional

$$L(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta}) := \int_\Omega \left[\frac{1}{2} \mathsf{C}\,\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\phi}) + \boldsymbol{\zeta} \cdot (\mathbf{grad}\,\omega - \boldsymbol{\phi}) - \frac{t^2}{2\lambda} |\boldsymbol{\zeta}|^2 - g\omega \right] d\boldsymbol{x},$$

which is defined on $\boldsymbol{H}^1(\Omega) \times H^1(\Omega) \times \boldsymbol{L}^2(\Omega)$. This is a mixed variational principle, and $(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta})$ is a saddle-point of L, not an extremum. Unlike the displacement variational formulation introduced earlier, the mixed formulation does not degenerate with small t, and makes sense even for $t = 0$. However, since L is not positive definite, not all choices of subspaces will lead to stable Galerkin discretizations. That is, if we determine an approximate solution $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\zeta}_h)$ by seeking a critical point of L in some finite-dimensional trial space, it need not be true that such a critical point exists, and even if it does, there may not be any constant for which the error in the approximate solution is at most that constant times the error in best approximation in that space. Mixed finite element spaces

must be chosen with regard to stability as well as approximation. Thus the passage to the mixed formulation does not in itself solve the problem of accurate uniform approximation, but it does fundamentally change the nature of the difficulty to be surmounted. Galerkin methods based on the displacement principle will automatically be stable, but will tend to lock. Galerkin methods based on the mixed principle, will tend to be uniform in t, but may not be stable.

Differentiating L we obtain a weakly formulated boundary value problem: the triple $(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta}) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ satisfies

$$
\begin{aligned}
a(\boldsymbol{\phi}, \boldsymbol{\psi}) - \int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\psi}\, d\boldsymbol{x} &= 0 \text{ for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega), \\
\int_\Omega \boldsymbol{\zeta} \cdot \mathbf{grad}\, \nu\, d\boldsymbol{x} &= \int_\Omega g\nu\, d\boldsymbol{x} \text{ for all } \nu \in \mathring{H}^1(\Omega), \\
\int_\Omega \left[(\mathbf{grad}\, \omega - \boldsymbol{\phi}) \cdot \boldsymbol{\eta}\, d\boldsymbol{x} - \lambda^{-1} t^2 \boldsymbol{\zeta} \cdot \boldsymbol{\eta} \right] d\boldsymbol{x} &= 0 \text{ for all } \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega),
\end{aligned}
$$

where $a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \int_\Omega \mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\psi})\, d\boldsymbol{x}$. The next key idea, introduced by Brezzi and Fortin in [6], is to decompose the shear force vector $\boldsymbol{\zeta}$ into it solenoidal and irrotational parts. That is, we introduce the Helmholtz decomposition

$$
\boldsymbol{\zeta} = \mathbf{grad}\, r + \mathbf{curl}\, p
$$

which determines $r \in \mathring{H}^1(\Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$ uniquely[1]. Analogously, we decompose the test function $\boldsymbol{\eta}$ as $\mathbf{grad}\, s + \mathbf{curl}\, q$. Introducing these decompositions into the weak formulation and using the orthogonality between gradients and curls we find that the quadruple $(r, \boldsymbol{\phi}, p, \omega) \in \mathring{H}^1(\Omega) \times \mathring{\boldsymbol{H}}^1(\Omega) \times H^1(\Omega)/\mathbb{R} \times \mathring{H}^1(\Omega)$ satisfies

$$
\begin{aligned}
\int_\Omega \mathbf{grad}\, r \cdot \mathbf{grad}\, \nu\, d\boldsymbol{x} &= \int_\Omega g\nu\, d\boldsymbol{x} \text{ for all } \nu \in \mathring{H}^1(\Omega), \\
a(\boldsymbol{\phi}, \boldsymbol{\psi}) - \int_\Omega \mathbf{curl}\, p \cdot \boldsymbol{\psi}\, d\boldsymbol{x} &= \int_\Omega \mathbf{grad}\, r \cdot \boldsymbol{\psi}\, d\boldsymbol{x} \text{ for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega), \\
\int_\Omega (-\boldsymbol{\phi} \cdot \mathbf{curl}\, q\, d\boldsymbol{x} - \lambda^{-1} t^2\, \mathbf{curl}\, p \cdot \mathbf{curl}\, q)\, d\boldsymbol{x} &= 0 \text{ for all } q \in H^1(\Omega)/\mathbb{R}, \\
\int_\Omega \mathbf{grad}\, \omega \cdot \mathbf{grad}\, s\, d\boldsymbol{x} = \int_\Omega (\boldsymbol{\phi} + \lambda^{-1} t^2\, \mathbf{grad}\, r) \cdot \mathbf{grad}\, s\, d\boldsymbol{x} &= 0 \text{ for all } \boldsymbol{s} \in \mathring{H}^1(\Omega).
\end{aligned}
\tag{3}
$$

The first equation, which decouples from the others, represents a weak formulation of the Dirichlet problem for Poisson's equation. This equation determines r. Once r is known, the next two equations form a coupled system for $\boldsymbol{\phi}$ and p. Finally, when these are known

[1] This decomposition is appropriate for the soft and hard clamped and soft and hard simply supported plates. For the free plate we instead take $r \in H^1(\Omega)/\mathbb{R}$ and $p \in \mathring{H}^1(\Omega)$. See [2], [3], [5].

the last equation determines ω as the solution of another Dirichlet problem for Poisson's equation. By making the change of variable $(\phi_1, \phi_2) \mapsto (-\phi_2, \phi_1)$, and similarly for $\boldsymbol{\psi}$, one easily recognizes the coupled system represented by the middle two equations as a singular perturbation of the Stokes system. It is, of course, no problem to stably discretize the two Poisson problems. For example, we may seek r_h and ω_h in the space of continuous piecewise linear functions, and allow the test functions s and ν to vary over the same space. The stability issue has thus been concentrated in the perturbed Stokes system. Fortunately approximation of the Stokes equations has been widely studied and there are many stable Stokes elements. Many of these methods do not immediately apply, however, because they use discontinuous approximation of the pressure variable p, and—due to the perturbation term—we need an H^1 approximation of it[2]. Thus it is natural to use a stable Stokes element with continuous pressure approximation to approximate $\boldsymbol{\phi}$ and p. There are a number of such elements known. The simplest, and one which has comparable accuracy to the piecewise linear approximation, is the MINI element of Arnold, Brezzi, and Fortin [1]. This element uses the space of continuous piecewise linear functions to approximate p and the same space enriched by a single bubble function on each triangle for each component of $\boldsymbol{\phi}$. The bubble function, which is usually chosen to be the cubic polynomial on the triangle which vanishes on the edges, can in fact be any function which is supported in the triangle and has non-vanishing integral. The bubble is used to ensure stability (without it we would have the linear-linear Stokes element, which is unstable), but doesn't aid approximation.

Let us denote by M_0^1 the space of continuous piecewise linear functions subordinate to a given mesh, by $\mathring{M}_0^1$ the subspace of such functions which vanish on the boundary, and by $\mathring{N}_0^1$ the sum of $\mathring{M}_0^1$ plus the span of one bubble function for each triangle. The method we have just derived is thus:

Problem BF: Find $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h) \in \mathring{M}_0^1 \times \mathring{\boldsymbol{N}}_0^1 \times \mathring{M}_0^1 \times \mathring{M}_0^1$ such that

$$\int_\Omega \mathbf{grad}\, r_h \cdot \mathbf{grad}\, \nu \, d\boldsymbol{x} = \int_\Omega g\nu \, d\boldsymbol{x} \text{ for all } \nu \in \mathring{M}_0^1,$$

$$a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - \int_\Omega \mathbf{curl}\, p_h \cdot \boldsymbol{\psi}) \, d\boldsymbol{x} = \int_\Omega \mathbf{grad}\, r_h \cdot \boldsymbol{\psi} \, d\boldsymbol{x} \text{ for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{N}}_0^1,$$

$$\int_\Omega \left[-\boldsymbol{\phi}_h \cdot \mathbf{curl}\, q \, d\boldsymbol{x} - \lambda^{-1} t^2 \, \mathbf{curl}\, p_h \cdot \mathbf{curl}\, q \right] \, d\boldsymbol{x} = 0 \text{ for all } q \in M_0^1/\mathbb{R},$$

$$\int_\Omega \mathbf{grad}\, \omega_h \cdot \mathbf{grad}\, s \, d\boldsymbol{x} = \int_\Omega (\boldsymbol{\phi}_h + \lambda^{-1} t^2 \, \mathbf{grad}\, r_h) \cdot \mathbf{grad}\, s \, d\boldsymbol{x} = 0 \text{ for all } \boldsymbol{s} \in \mathring{M}_0^1.$$

This method is precisely the method proposed by Brezzi and Fortin in [6]. One can prove that it indeed provides optimal order approximation.

[2]Actually, there is a way to use discontinuous elements for p. We can introduce $\mathbf{curl}\, p$ as yet another dependent variable to be approximated in $H(\mathrm{rot})$ independently of $p \in L^2$. Thus the Δp perturbation term would itself be treated by a mixed method. This approach leads to a number of interesting methods studied recently by Bathe, Brezzi, and Fortin.

THEOREM. *Problem BF admits a unique solution. Moreover there is a constant C independent of the mesh size h and the plate thickness t such that*

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1+\|\omega-\omega_h\|_1\le Ch\|g\|_0,\quad \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0+\|\omega-\omega_h\|_0\le Ch^2\|g\|_0.$$

The proof is essentially straightforward. The first estimate was proven in [6] and the arguments of [4] can be easily adapted to give the second. Thus this method can be said to solve the problem of finding a low order uniformly optimal approximation of the Reissner–Mindlin system. However its form is quite different from the original variational formulation given in terms of the displacement variables $\boldsymbol{\phi}$ and ω only. From a practical point of view, it is valuable to have a method with similar convergence properties which can be computed in the primitive variables. In order to motivate our modification of the Brezzi–Fortin method, let us eliminate, formally, the additional variables from Problem BF. First we define $\boldsymbol{\zeta}_h=\mathbf{grad}\,r_h+\mathbf{curl}\,p_h$ which belongs to the space $\boldsymbol{L}_h:=\mathbf{grad}\,\mathring{M}_0^1+\mathbf{curl}\,M_0^1$. It is then easy to see that $\boldsymbol{\phi}_h$, ω_h, and $\boldsymbol{\zeta}_h$ are determined from the mixed method:

Find $(\boldsymbol{\phi}_h,\omega_h,\boldsymbol{\zeta}_h)\in\mathring{N}_0^1\times\mathring{M}_0^1\times\boldsymbol{L}_h$ satisfying

$$\begin{aligned}
a(\boldsymbol{\phi}_h,\boldsymbol{\psi})-\int_\Omega\boldsymbol{\zeta}_h\cdot\boldsymbol{\psi}\,d\boldsymbol{x}&=0\ \text{for all }\boldsymbol{\psi}\in\mathring{N}_0^1,\\
\int_\Omega\boldsymbol{\zeta}_h\cdot\mathbf{grad}\,\nu\,d\boldsymbol{x}&=\int_\Omega g\nu\,d\boldsymbol{x}\ \text{for all }\nu\in\mathring{M}_0^1,\\
\int_\Omega\left[(\mathbf{grad}\,\omega_h-\boldsymbol{\phi}_h)\cdot\boldsymbol{\eta}\,d\boldsymbol{x}-\lambda^{-1}t^2\boldsymbol{\zeta}_h\cdot\boldsymbol{\eta}\right]d\boldsymbol{x}&=0\ \text{for all }\boldsymbol{\eta}\in\boldsymbol{L}_h.
\end{aligned}\tag{4}$$

Next we can eliminate $\boldsymbol{\zeta}_h$ from this formulation and obtain the primitive variables directly from the displacement method:

Find $(\boldsymbol{\phi}_h,\omega_h)\in\mathring{N}_0^1\times\mathring{M}_0^1$ satisfying

$$\begin{aligned}
a(\boldsymbol{\phi}_h,\boldsymbol{\psi})+\lambda t^{-2}\int_\Omega\boldsymbol{P}_{\boldsymbol{L}_h}(\mathbf{grad}\,\omega_h-\boldsymbol{\phi}_h)\cdot(\mathbf{grad}\,\nu-\boldsymbol{\psi})\,d\boldsymbol{x}&\\
=\int_\Omega g\nu\,d\boldsymbol{x}\ \text{for all }(\boldsymbol{\psi},\nu)\in\mathring{N}_0^1\times\mathring{M}_0^1,&
\end{aligned}\tag{5}$$

where $\boldsymbol{P}_{\boldsymbol{L}_h}$ denotes the orthogonal projection of $\boldsymbol{L}^2$ into $\boldsymbol{L}_h$. Note that if this projection is suppressed this becomes a standard displacement method: (5) would then just be the statement that $(\boldsymbol{\phi}_h,\omega_h)$ minimizes the energy functional J over $\mathring{N}_0^1\times\mathring{M}_0^1$. With the projection, we call (5) a *generalized displacement method*. It is equivalent to the Brezzi–Fortin mixed method in that the values of $\boldsymbol{\phi}_h$ and ω_h computed from Problem BF and from (5) are are exactly the same. In a sense, the generalized displacement method is a method in the primitive variables which is uniformly optimal order accurate. It is not practical to implement, however, because the projection operator $\boldsymbol{P}_{\boldsymbol{L}_h}$ is difficult to compute.

3. A uniformly accurate method in the primitive variables. The next idea which we apply is the discrete Helmholtz decomposition introduced by Arnold and Falk in [4]. To state it we introduce some notation. Let $\boldsymbol{M}^0_{-1}$ denote the space of piecewise constant vectorfields. Let M^1_* denote the non-conforming piecewise linear approximation of $H^1(\Omega)$, i.e., the space of all functions which are linear over each triangle of the triangulation and which are continuous at the midpoints of interelement edges, and let $\mathring{M}^1_*$ be the subspace of such functions which vanish at the midpoints of boundary edges. Finally, For $r \in M^1_*$ we denote by $\mathbf{grad}_h\, r$ the gradient of r taken piecewise, which is a funtion in $\boldsymbol{M}^0_{-1}$. The discrete Helmholtz decomposition, which can be verified easily, states that every vectorfield in $\boldsymbol{M}^0_{-1}$ can be decomposed as $\mathbf{grad}_h\, s + \mathbf{curl}\, q$ for some unique $s \in \mathring{M}^1_*$ and $q \in M^1_0/\mathbb{R}$. Moreover, this is an L^2 orthogonal decomposition.

In light of the discrete Helmholtz decomposition and the considerations which led to Problem BF, it is natural to discretize (3) using $\mathring{N}^1_0$ for $\boldsymbol{\phi}$ and $M^1_0/\mathbb{R}$ for p (the MINI element), and then $\mathring{M}^1_*$ for r. Similarly the test functions $\boldsymbol{\psi}$, q, and s should range over $\mathring{N}^1_0$, $M^1_0/\mathbb{R}$, and $\mathring{M}^1_*$ respectively. Since r is to be approximated in $\mathring{M}^1_*$, the first equation of (3) suggests that the test function ν should vary over the same space. Similarly, since s is to vary over $\mathring{M}^1_*$, the last equation of (3) leads us to approximate $\omega \in \mathring{M}^1_*$. Thus we have obtained the following method:

Problem AF: Find $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h) \in \mathring{M}^1_* \times \mathring{\boldsymbol{N}}^1_0 \times \mathring{M}^1_0 \times \mathring{M}^1_*$ such that

$$\int_\Omega \mathbf{grad}_h\, r_h \cdot \mathbf{grad}_h\, \nu \, d\boldsymbol{x} = \int_\Omega g\nu \, d\boldsymbol{x} \text{ for all } \nu \in \mathring{M}^1_*,$$

$$a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - \int_\Omega \mathbf{curl}\, p_h \cdot \boldsymbol{\psi} \, d\boldsymbol{x} = \int_\Omega \mathbf{grad}_h\, r \cdot \boldsymbol{\psi} \, d\boldsymbol{x} \text{ for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{N}}^1_0,$$

$$\int_\Omega \left[-\boldsymbol{\phi}_h \cdot \mathbf{curl}\, q \, d\boldsymbol{x} - \lambda^{-1} t^2 \, \mathbf{curl}\, p_h \cdot \mathbf{curl}\, q \right] d\boldsymbol{x} = 0 \text{ for all } q \in M^1_0/\mathbb{R},$$

$$\int_\Omega \mathbf{grad}_h\, \omega_h \cdot \mathbf{grad}_h\, s \, d\boldsymbol{x} = \int_\Omega (\boldsymbol{\phi}_h + \lambda^{-1} t^2 \, \mathbf{grad}_h\, r_h) \cdot \mathbf{grad}_h\, s \, d\boldsymbol{x} = 0 \text{ for all } \boldsymbol{s} \in \mathring{M}^1_*.$$

The only difference between this method and that of Brezzi and Fortin is the use of non-conforming linear elements for the Poisson equations. In fact these elements are known to converge with optimal order, and it is not hard to establish the uniform convergence of our method [4].

THEOREM. *Problem AF admits a unique solution. Moreover there is a constant C independent of the mesh size h and the plate thickness t such that*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\, \mathbf{grad}_h(\omega - \omega_h)\|_0 \le Ch\|g\|_0, \quad \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^2\|g\|_0.$$

While the introduction of non-conforming elements does not effect the convergence of the method, it has profound effect on the implementation. If we introduce $\boldsymbol{\zeta}_h = \mathbf{grad}_h\, r +$

$\mathbf{curl}\, p$ as before, then we find that $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\zeta}_h)$ satisfy (4) with the space $\boldsymbol{L}_h$ replaced by $\mathbf{grad}_h \mathring{M}_0^1 + \mathbf{curl}\, M_0^1$. By the discrete Helmholtz decomposition this is equal to $\boldsymbol{M}_{-1}^0$, the space of all piecewise constant vectorfields. Consequently, when we eliminate $\boldsymbol{\zeta}_h$ to obtain a generalized displacement method, we find that $(\boldsymbol{\phi}_h, \omega_h) \in \mathring{N}_0^1 \times \mathring{M}_*^1$ satisfies

$$(6)\quad a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2} \int_\Omega \boldsymbol{P}_0(\mathbf{grad}\, \omega_h - \boldsymbol{\phi}_h) \cdot (\mathbf{grad}\, \nu - \boldsymbol{\psi})\, d\boldsymbol{x} = \int_\Omega g\nu\, d\boldsymbol{x} \text{ for all } (\boldsymbol{\psi}, \nu) \in \mathring{N}_0^1 \times \mathring{M}_*^1,$$

where $\boldsymbol{P}_0$ denotes the orthogonal projection of $\boldsymbol{L}^2$ into $\boldsymbol{M}_{-1}^0$. This method appears very similar to (5), the only differences being the use of the non-conforming space $\mathring{M}_*^1$ in place of $\mathring{M}_0^1$ and the use of the projection $\boldsymbol{P}_0$ instead of $\boldsymbol{P}_{\boldsymbol{L}_h}$. This latter difference is of great significance however, because, unlike $\boldsymbol{P}_{\boldsymbol{L}_h}$, $\boldsymbol{P}_0$ can be computed inexpensively during the usual finite element assembly process. In fact on every triangle T, $\boldsymbol{P}_0\boldsymbol{\eta}$ is simply the average value of $\boldsymbol{\eta}$ on T.

4. Elimination of the bubble functions. The bubble functions, which were introduced into the trial space for $\boldsymbol{\phi}$ for stability reasons, can be eliminated by the process of static condensation. For each triangle T, let b_T denote the normalized bubble function on T, i.e., the unique cubic polynomial in $\mathring{H}^1(T)$ with average value 1. Set $\boldsymbol{b}_T^1 = (b_T, 0)$ and $\boldsymbol{b}_T^2 = (0, b_T)$. Then $\boldsymbol{\phi}_h \in \mathring{N}_0^1$ can be written as

$$\boldsymbol{\phi}_h = \bar{\boldsymbol{\phi}}_h + \sum_T (c_T^1 \boldsymbol{b}_T^1 + c_T^2 \boldsymbol{b}_T^2)$$

for a unique $\bar{\boldsymbol{\phi}}_h \in \mathring{M}_0^1$ (the piecewise linear interpolant of $\boldsymbol{\phi}_h$) and unique constants c_T^i. Now $a(\bar{\boldsymbol{\phi}}_h, \boldsymbol{b}_T^i) = 0$, as follows from integration by parts, $\int_\Omega \boldsymbol{P}_0 \boldsymbol{b}_T^i \cdot \boldsymbol{b}_{T'}^j\, d\boldsymbol{x} = 0$ if $T \neq T'$, and $\int_\Omega \boldsymbol{P}_0 \boldsymbol{b}_T^i \cdot \boldsymbol{b}_T^j\, d\boldsymbol{x} = |T|\delta_{ij}$ with $|T|$ denoting the measure of T. Therefore, setting $\nu = 0$, $\boldsymbol{\psi} = \boldsymbol{b}_T^i$ in (6), we obtain

$$c_T^1 a(\boldsymbol{b}_T^1, \boldsymbol{b}_T^i) + c_T^2 a(\boldsymbol{b}_T^2, \boldsymbol{b}_T^i) + \lambda |T| t^{-2} (c_T^1 \delta_{1i} + c_T^2 \delta_{2i}) = \lambda |T| t^{-2} (\mathbf{grad}_h\, \omega_h - \boldsymbol{P}_0 \bar{\boldsymbol{\phi}}_h)|_T \cdot (\delta_{1i}, \delta_{2i}).$$

Setting

$$\boldsymbol{c}_T = \begin{pmatrix} c_T^1 \\ c_T^2 \end{pmatrix}, \quad \mathcal{B}_T = \begin{pmatrix} a(\boldsymbol{b}_T^1, \boldsymbol{b}_T^1) & a(\boldsymbol{b}_T^2, \boldsymbol{b}_T^1) \\ a(\boldsymbol{b}_T^1, \boldsymbol{b}_T^2) & a(\boldsymbol{b}_T^2, \boldsymbol{b}_T^2) \end{pmatrix},$$

this equation becomes

$$(\mathcal{B}_T + \lambda |T| t^{-2} \mathcal{I}) \boldsymbol{c}_T = \lambda |T| t^{-2} (\mathbf{grad}_h\, \omega_h - \boldsymbol{P}_0 \bar{\boldsymbol{\phi}}_h)|_T,$$

or

$$(7)\qquad \boldsymbol{c}_T = \left(\mathcal{I} + \lambda^{-1} |T|^{-1} t^2 \mathcal{B}_T\right)^{-1} (\mathbf{grad}_h\, \omega_h - \boldsymbol{P}_0 \bar{\boldsymbol{\phi}}_h)|_T.$$

Let us set $\mathcal{A}_T = \lambda|T|h_T^{-2}\mathcal{B}_T^{-1}$ where h_T is the diameter of T. Then

$$t^{-2}[\mathcal{I} - (\mathcal{I} + \lambda^{-1}|T|^{-1}t^2\mathcal{B}_T)^{-1}] = (t^2\mathcal{I} + h_T^2\mathcal{A}_T)^{-1}. \tag{8}$$

We now take $\boldsymbol{\psi} \in \mathring{M}_0^1$ and $\nu \in \mathring{M}_*^1$ in (6) and apply (7) and (8). It follows that $(\bar{\boldsymbol{\phi}}_h, \omega_h) \in \mathring{M}_0^1 \times \mathring{M}_*^1$ satisfies

$$a(\bar{\boldsymbol{\phi}}_h, \boldsymbol{\psi}) + \lambda \sum_T (t^2\mathcal{I} + h_T^2\mathcal{A}_T)^{-1} \int_T \boldsymbol{P}_0(\mathbf{grad}\,\omega_h - \bar{\boldsymbol{\phi}}_h) \cdot (\mathbf{grad}\,\nu - \boldsymbol{\psi})\, d\boldsymbol{x} = \int_\Omega g\nu\, d\boldsymbol{x} \text{ for all } (\boldsymbol{\psi}, \nu) \in \mathring{M}_0^1 \times \mathring{M}_*^1. \tag{9}$$

In this generalized displacement method the elements are strictly linear, no bubbles, and in addition to the addition of the averaging projection $\boldsymbol{P}_0$, the coefficient t^{-2} has been replaced by $(t^2\mathcal{I} + h_T^2\mathcal{A}_T)^{-1}$. Note that $\mathbf{grad}\,\omega_h - \bar{\boldsymbol{\phi}}_h$ is a piecewise linear function, so that on T, $\boldsymbol{P}_0(\mathbf{grad}\,\omega_h - \bar{\boldsymbol{\phi}}_h)$ is simply the value of this function at the barycenter of T. Therefore the projection can be implemented simply by using a one-point quadrature on the relevant term.

5. Connections with other methods. The passage from the generalized displacement method given in (6) to that in (9) involves dropping the bubbles from the rotation space and simultaneously replacing t^2 with $t^2 + h_T^2\mathcal{A}_T$. This is very reminiscent of the stabilization method introduced by Brezzi and Pitkäranta in [7] for the Stokes equation, and subsequently generalized and applied to other problems in many papers of which we mention [10], in particular. From this point of view the particular matrix $\mathcal{A}_T$ appearing in (9) is not important—any positive definite matrix with positive upper and lower bounds on its eigenvalues may be used. Motivated by such stabilization approaches, two groups of researchers working independently, Duran, Ghioldi, and Wolanksi [8], and Franca and Stenberg [9], recently proposed a method which is identical to (9) except that they take $\mathcal{A}_T$ to be an arbitrary positive scalar. Each of these groups proved uniform optimal order convergence of the resulting method. Since it is quite clear that the assumption that $\mathcal{A}_T$ is scalar plays no fundamental role, one may view (9) as a special case of the method proposed in [8] and [9]. It seems unlikely that the particular value of $\mathcal{A}_T$ which arises from elimination of the bubbles is in any sense optimal. In fact, by replacing b_T with another function of average value 1 which vanishes on ∂T, we will get a variant of (6) with the same convergence properties, but the value of $\mathcal{A}_T$ which arises after static condensation will be different. In the context of the Stokes equations, Pierre [11] has compared computationally the use of bubbles to stabilize linear velocity–linear pressure elements (giving rise to the MINI element) to the stabilization method of [7] and he found no particular advantage to the former over the latter. Thus it would seem that preferred form for computation is (9) with $\mathcal{A}_T$ chosen in a convenient manner (perhaps after some experimentation).

Let us mention that the stabilization procedure in [7] can be applied to the generalized displacement method (6) as just discussed, to the corresponding mixed method involving $\boldsymbol{\phi}_h$, ω_h, and $\boldsymbol{\zeta}_h$, or to the formulation of Brezzi–Fortin involving $\boldsymbol{\phi}_h$, ω_h, r_h and p_h. The latter approach is very close indeed to the original idea in [7]: we first reduce the Reissner–Mindlin system to some Poisson equations and the Stokes equations, and then we use the method proposed in [7] to solve the Stokes equations. When non-conforming linear elements are used for the Poisson equation, one can use the discrete Helmholtz decomposition to show that all three approaches yield equivalent methods. In [12] Pierre considered the use of this stabilization approach to remove the bubble functions from the method (3) of Brezzi and Fortin. Of course for that method the discrete Helmholtz decomposition doesn't apply, and there is no practical way to compute it in the primitive variables.

6. List of notations. Boldface type is used to denote vector quantities, including vector-valued functions and operators whose values are vector-valued functions. Script type is used to denote matrix quantities and sans serif type is used for higher order tensor quantities.

$a(\boldsymbol{\phi},\boldsymbol{\psi})$	$= \int_\Omega \mathsf{C}\,\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\psi})\,d\boldsymbol{x}$
C	tensor of bending moduli
curl	(vector) curl of a scalar function
div	divergence of a vector function
div	divergence of a matrix function
E	Young's modulus
g	scaled loading function, = transverse load density per unit area divided by t^3
grad	gradient of a scalar function
$\mathbf{grad}_h$	piecewise gradient of a piecewise smooth function
$\mathcal{GRAD}$	gradient of a vector function
$H^1(\Omega)$	the Sobolev space of L^2 functions with square integrable derivatives
$\mathring{H}^1(\Omega)$	the space of $H^1(\Omega)$ functions vanishing on $\partial\Omega$
$L^2(\Omega)$	the Lebesgue space of square integrable functions on Ω
$\mathcal{I}$	2×2 identity matrix
k	shear correction factor
$\boldsymbol{L}_h$	$\mathbf{grad}\,\mathring{M}_0^1 + \mathbf{curl}\,M_0^1$
M_{-1}^0	finite element space of piecewise constant functions
M_0^1	finite element space of continuous piecewise linear functions
$\mathring{M}_0^1$	$M_0^1 \cap \mathring{H}^1(\Omega)$
$\mathring{N}_0^1$	space spanned by M_0^1 and one bubble function per triangle
$\mathring{M}_*^1$	non-conforming piecewise linear approximation of $\mathring{H}^1(\Omega)$

$\boldsymbol{P}_0$	orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{M}^0_{-1}$
$P_{\boldsymbol{L}_h}$	orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{L}_h$
p	potential for solenoidal part of $\boldsymbol{\zeta}$
r	potential for irrotational part of $\boldsymbol{\zeta}$
t	plate thickness
$\mathcal{E}$	(matrix) symmetric part of the gradient of a vector function
$\boldsymbol{\zeta}$	shear force vector= $\mathbf{grad}\, r + \mathbf{curl}\, p$
λ	$= Ek/[2(1+\nu)]$
ν	Poisson ratio
$\boldsymbol{\phi}$	rotation vector
ω	transverse displacement
Ω	region occupied by the midplane of the plate

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