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Some New Elements for the Reissner–Mindlin Plate Model

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Introduction

In this work-in-progress we report on a new approach to obtaining stable locking-free discretizations of the Reissner–Mindlin plate model. For a plate of thickness t with midplane section $\Omega \subset \mathbb{R}^2$ the clamped Reissner–Mindlin plate model determines ω, the transverse displacement of the midplane, and $\boldsymbol{\phi}$, the rotation of fibers normal to the midplane, as the unique minimizer over $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$ of the energy functional:

$$J(\boldsymbol{\phi}, \omega) := \frac{1}{2}\,(\mathcal{C}\,\mathcal{E}\,\boldsymbol{\phi}, \mathcal{E}\,\boldsymbol{\phi}) + \frac{\lambda t^{-2}}{2}\|\boldsymbol{\phi} - \mathbf{grad}\,\omega\|_0^2 - (g, \omega), \tag{1}$$

where gt^3 is the transverse load force density per unit area, $\mathcal{E}\,\boldsymbol{\phi}$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, λ is a constant, and $\mathcal{C}$ is a positive-definite fourth order tensor, with λ and $\mathcal{C}$ both depending on Young's modulus E and the Poisson ratio ν. We use parentheses to denote the $L^2(\Omega)$ innerproduct and $\|\cdot\|_s$ to denote the norm in the Sobolev space $H^s(\Omega)$.

This variational formulation may be discretized by finite elements in a standard way, but such discretizations typically lock, that is, the approximation they afford is not uniform in t, but deteriorates as t tends to 0. A mixed formulation is derived by introducting the scaled shear stress

$$\boldsymbol{\zeta} = \lambda t^{-2}(\boldsymbol{\phi} - \mathbf{grad}\,\omega)$$

as an independent unknown. The triple $(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta})$ can then be characterized as the unique critical point of the functional

$$L(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta}) := \frac{1}{2}(\mathcal{C}\,\mathcal{E}\,\boldsymbol{\phi}, \mathcal{E}\,\boldsymbol{\phi}) + (\boldsymbol{\phi} - \mathbf{grad}\,\omega, \boldsymbol{\zeta}) - \frac{t^2}{2\lambda}\|\boldsymbol{\zeta}\|_0^2 - (g, \omega),$$

on $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$. Note that the functional L, unlike J, is not positive definite and $(\boldsymbol{\phi}, \omega, \boldsymbol{\zeta})$ is a saddle-point not an extremum. Consequently, not all choices of subspaces will lead to stable Galerkin discretizations. That is, if we determine an approximate solution ($\boldsymbol{\phi}_h$, ω_h, $\boldsymbol{\zeta}_h$) by seeking a critical point of L in some finite-dimensional trial space, it need not be true that such a critical point exists, and even if it does it may not give quasioptimal approximation. On the other hand, in contrast to the previous displacement variational formulation, the mixed formulation does not degenerate with small t, and makes sense even for $t = 0$. Thus the passage to the mixed formulation does not in itself solve the problem of accurate uniform approximation, but it does fundamentally change the nature of the difficulty to be surmounted. Finite element methods based on the displacement principle, although always stable, tend to lock, while finite element methods based on the mixed principle tend to be uniform in t, but may not be stable.

Let $b : [\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)] \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ be defined by

$$b(\boldsymbol{\psi}, \mu; \boldsymbol{\eta}) = (\boldsymbol{\psi} - \mathbf{grad}\,\mu, \boldsymbol{\eta}).$$

A choice $\boldsymbol{V}_h \subset \mathring{\boldsymbol{H}}^1(\Omega)$, $W_h \subset \mathring{H}^1(\Omega)$, $\boldsymbol{S}_h \subset \boldsymbol{L}^2(\Omega)$ of subspaces for discretization of the mixed formulation will be stable if two conditions are met:

S1. There exists a bounded linear operator $\Pi_h : \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega) \to \boldsymbol{V}_h \times W_h$, for which

$$b(\Pi_h(\boldsymbol{\psi}, \mu); \boldsymbol{\eta}) = b(\boldsymbol{\psi}, \mu; \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{S}_h.$$

S2. If $(\boldsymbol{\psi}, \mu) \in \boldsymbol{V}_h \times W_h$ and $b(\boldsymbol{\psi}, \mu; \boldsymbol{\eta}) = 0$ for all $\boldsymbol{\eta} \in \boldsymbol{S}_h$, then $\boldsymbol{\psi} = \mathbf{grad}\,\mu$.

In this case it can be shown by standard arguments that there exists a constant C independent of t and h (but depending on the bound for Π_h in condition S1) such that

$$(2) \quad \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1+\|\omega-\omega_h\|_1+|||\boldsymbol{\zeta}-\boldsymbol{\zeta}_h|||_t \le C \inf(\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_1+\|\omega-\mu\|_1+|||\boldsymbol{\zeta}-\boldsymbol{\eta}|||_t).$$

Here the infimum is taken over all $\boldsymbol{\psi} \in \boldsymbol{V}_h$, $\mu \in W_h$, and $\boldsymbol{\eta} \in \boldsymbol{S}_h$, and

$$|||\boldsymbol{\eta}|||_t := \|\boldsymbol{\eta}\|_{-1} + \|\operatorname{div}\boldsymbol{\eta}\|_{-1} + t\|\boldsymbol{\eta}\|_0.$$

Condition S1 is essentially Fortin's criterion [4, §II.2.3] for verifying the inf-sup condition associated with the bilinear form b. It insures that the space

$\boldsymbol{V}_h \times W_h$ is sufficiently large with respect to $\boldsymbol{S}_h$ to give control over $\boldsymbol{\zeta}_h$. Notice that condition S1 becomes easier to satisfy as $\boldsymbol{V}_h$ and W_h become larger and $\boldsymbol{S}_h$ becomes smaller, while the opposite is true for condition S2. This opposition contributes to the difficulty in selecting spaces which satisfy both. Indeed, very few practical choices of spaces have been shown to satisfy S1 and S2. We now show how a slight modification to the mixed formulation eliminates the need for condition S2, thereby considerably simplifying the choice of finite element spaces. Based on this we will present several choices in the following sections.

For the modified mixed method we take $\boldsymbol{\zeta} = \lambda(t^{-2}-1)(\boldsymbol{\phi} - \mathbf{grad}\,\omega)$ and set

$$a(\boldsymbol{\phi}, \omega; \boldsymbol{\psi}, \mu) := (\mathcal{C}\,\mathcal{E}\,\boldsymbol{\phi}, \mathcal{E}\,\boldsymbol{\psi}) + \lambda(\boldsymbol{\phi} - \mathbf{grad}\,\omega, \boldsymbol{\psi} - \mathbf{grad}\,\mu),$$

which defines a bounded bilinear form on $[\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)] \times [\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)]$. Then we have

$$a(\boldsymbol{\phi}, \omega; \boldsymbol{\psi}, \mu) + b(\boldsymbol{\psi}, \mu; \boldsymbol{\zeta}) = (g, \mu), \quad (\boldsymbol{\psi}, \mu) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega),$$
$$b(\boldsymbol{\phi}, \omega; \boldsymbol{\eta}) - \frac{t^2}{\lambda(1-t^2)}(\boldsymbol{\zeta}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega),$$

which is the weak form of a modified saddle point problem.

Using Korn's inequality and Poincare's inequality it is not difficult to show that that there is a positive constant c for which

$$a(\boldsymbol{\psi}, \mu; \boldsymbol{\psi}, \mu) \geq c(\|\boldsymbol{\psi}\|_1^2 + \|\mu\|_1^2) \quad \text{for all } (\boldsymbol{\psi}, \mu) \in \boldsymbol{H}^1(\Omega) \times \mathring{H}^1(\Omega).$$

Because the bilinear form a in this modified mixed method is positive definite on all of $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$, condition S2 is not necessary for stability of discretizations. (For a related idea in a different context, see [5].) If the spaces $\boldsymbol{V}_h$, W_h, and $\boldsymbol{S}_h$ are chosen satisfying condition S1, then the quasioptimal estimate (2) follows. Although the simplest choices of finite element spaces probably don't satisfy S1, in the next two sections we show how for some such choices $\boldsymbol{V}_h$ and W_h can be enriched using bubble degrees of freedom in order to achieve stability.

An element with piecewise constant shear approximation

Suppose that Ω is a polygon and $\mathcal{T}_h$ is a regular quasiuniform family of triangulations of Ω indexed by the mesh length parameter h tending to 0. Let $L_k^s(\mathcal{T}_h)$ denote the usual space of functions in $H^s(\Omega)$ which are piecewise polynomials of degree at most k with respect to the triangulation, and set $\mathring{L}_k^s(\mathcal{T}_h) = L_k^s(\mathcal{T}_h) \cap \mathring{H}^1(\Omega)$, and

$$B_k(\mathcal{T}_h) = \left\{\, v \in L_k^0(\mathcal{T}_h) \,|\, v|_{\partial T} = 0 \quad \text{for all } T \in \mathcal{T}_h \,\right\}.$$

We continue to use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. Script type is used in a similar way for 2×2-matrix objects. Thus for example if $\pi : L^2(\Omega) \to L^1_1(\mathcal{T}_h)$ is some operator, then $\boldsymbol{\pi}$ denotes the operator $\pi\times\pi : \boldsymbol{L}^2(\Omega)\to \boldsymbol{L}^1_1(\mathcal{T}_h)$.

We shall now verify the stability condition S1 for the choice of elements

$$\boldsymbol{V}_h = \mathring{\boldsymbol{L}}^1_1(\mathcal{T}_h) + \boldsymbol{B}_3(\mathcal{T}_h), \quad W_h = \mathring{L}^1_2(\mathcal{T}_h), \quad \boldsymbol{S}_h = \boldsymbol{L}^0_0, \tag{3}$$

indicated in Figure 1.

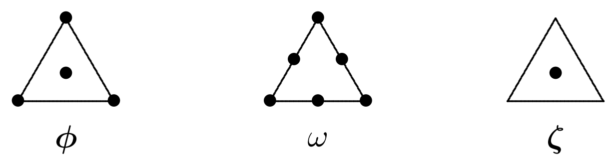

Fig. 1 Degrees of freedom for the finite element spaces in (3).

Note that the space $\boldsymbol{V}_h$ is exactly the one used in [1] to approximate the velocity for Stokes flow, while the pairing of piecewise quadratics and piecewise constants provides the well-known $P^2 - P^0$ Stokes element. We construct Π_h employing the ideas used to construct the corresponding operators for these Stokes elements. For details on these constructions see [4, §VI.4]. Let $\pi^0_h : \mathring{H}^1(\Omega) \to \mathring{L}^1_1(\mathcal{T}_h)$ denote an approximation operator satisfying

$$\|v - \pi^0_h v\|_s \le Ch^s \|v\|_1, \quad s = 0, 1, \quad v \in \mathring{H}^1(\Omega).$$

Define $\pi^1_h : \mathring{H}^1(\Omega) \to B_3(\mathcal{T}_h)$ and $\pi^2_h : \mathring{H}^1(\Omega) \to \mathring{L}^1_2(\mathcal{T}_h)$ by imposing

$$\int_T (v - \pi^1_h v) = 0, \quad \pi^2_h v(z) = \int_e (v - \pi^2_h v) = 0$$

for all triangles $T \in \mathcal{T}_h$, all vertices z of such triangles, and all edges e of such triangles. It then follows that the operators $\boldsymbol{\Pi}^1_h v := \boldsymbol{\pi}^0_h \boldsymbol{v} + \boldsymbol{\pi}^1_h(\boldsymbol{v} - \boldsymbol{\pi}^0_h \boldsymbol{v})$ and $\Pi^2_h v := \pi^0_h v + \pi^2_h (v - \pi^0_h v)$ map $\mathring{\boldsymbol{H}}^1(\Omega)$ into $\boldsymbol{V}_h$ and $\mathring{H}^1(\Omega)$ into W_h, respectively, that they are bounded in H^1 norm uniformly in h, and that

$$b(\boldsymbol{\Pi}^1_h \boldsymbol{\psi}, \Pi^2_h \mu; \boldsymbol{\zeta}) = b(\boldsymbol{\psi}, \mu; \boldsymbol{\zeta}), \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega), \mu \in \mathring{H}^1(\Omega), \boldsymbol{\zeta} \in \boldsymbol{S}_h.$$

Thus S1 holds with $\Pi_h(\boldsymbol{\psi}, \mu) = (\boldsymbol{\Pi}^1_h \boldsymbol{\psi}, \Pi^2_h \mu)$.

The discrete problem can be written in weak form as

Find $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\zeta}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{S}_h$ such that

$$a(\boldsymbol{\phi}_h, \omega_h; \boldsymbol{\psi}, \mu) + b(\boldsymbol{\psi}, \mu; \boldsymbol{\zeta}_h) = (g, \mu), \quad (\boldsymbol{\psi}, \mu) \in \boldsymbol{V}_h \times W_h,$$

$$b(\boldsymbol{\phi}_h, \omega_h; \boldsymbol{\eta}) - \frac{t^2}{\lambda(1-t^2)}(\boldsymbol{\zeta}_h, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{S}_h.$$

In view of the stability, we obtain the quasioptimal error estimate (2). It then follows immediately that

$$(4) \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\omega - \omega_h\|_1 + |||\boldsymbol{\zeta} - \boldsymbol{\zeta}_h|||_t \le Ch(\|\boldsymbol{\phi}\|_2 + \|\omega\|_2 + \|\boldsymbol{\zeta}\|_1)$$

with C independent of h and t. Using the Helmholtz decomposition and discrete Helmholtz decomposition as in [2], in place of $\|\boldsymbol{\zeta}\|_1$ in this estimate we may obtain the smaller quantity $\|\boldsymbol{\zeta}\|_0 + \|\operatorname{div}\boldsymbol{\zeta}\|_0 + t\|\boldsymbol{\zeta}\|_1$. This is preferable since, as can be deduced from Theorem 7.1 of [2], this quantity is bounded by $C\|g\|_0$ with C independent of t, while $\|\boldsymbol{\zeta}\|_1$ can be expected to grow unboundedly as $t \to 0$. (In any case, however, such regularity while valid for the clamped plate, is not true for some other important boundary conditions. Cf. [3].)

A drawback of this choice of elements, essentially inherited from the P^2–P^0 Stokes element, is that although we use quadratic approximation for the transverse displacement ω, we obtain only first order convergence in H^1 due to the lower degree approximation used for the other variables. The degrees of polynomial interpolation used for the different variables are in this sense not balanced.

Elements with continuous piecewise linear shear approximation

In this section we will present some choices of elements with continuous piecewise linear approximation of the shear stress $\boldsymbol{\zeta}$. Throughout the section we consider the (soft) simply supported Reissner–Mindlin plate, in which the energy function (1) is minimized over $\boldsymbol{H}^1(\Omega) \times \mathring{H}^1(\Omega)$, rather than the clamped plate discussed heretofore. The reason for this will be discussed at the end of the section.

In the simplest element of this sort we choose continuous piecewise linear interpolation for the rotation and shear stress variables and use continuous piecewise linear interpolation enriched by cubic bubbles for the transverse displacement. That is, we select

$$(5) \qquad \boldsymbol{V}_h = \boldsymbol{S}_h = \boldsymbol{L}_1^1(\mathcal{T}_h), \quad W_h = L_1^1(\mathcal{T}_h) + B_3(\mathcal{T}_h),$$

as shown in Figure 2.

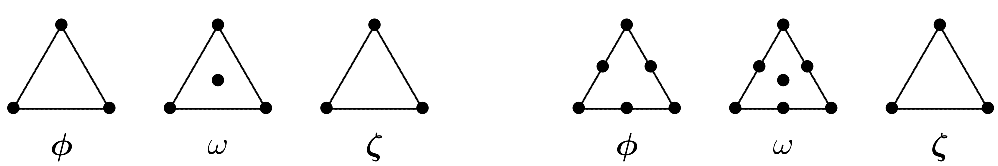

Fig. 2 Degrees of freedom for the finite element spaces in (5) and (7).

The operator Π_h^1 defined in the previous section maps $\mathring{H}^1(\Omega)$ onto W_h and satisfies $\int_T(\mu - \Pi_h^1\mu) = 0$ for all $\mu \in \mathring{H}^1(\Omega)$ and all $T \in \mathcal{T}_h$. Consequently

$$\big(\mathbf{grad}(\mu - \Pi_h^1\mu), \boldsymbol{\eta}\big) = -(\mu - \Pi_h^1\mu, \operatorname{div}\boldsymbol{\eta}) = 0$$

for all $\mu \in \mathring{H}^1(\Omega)$ and all $\boldsymbol{\eta} \in \boldsymbol{S}_h$. Let P_h^1 denote the $\boldsymbol{L}^2$-projection onto $L_1^1(\mathcal{T}_h)$. Under the assumption that the mesh family is quasiuniform as $h \to 0$, P_h^1 is bounded uniformly as an operator $H^1(\Omega) \to H^1(\Omega)$. Morever, since $\boldsymbol{S}_h \subset \boldsymbol{V}_h$ (in fact they are equal),

$$(\boldsymbol{\psi} - \boldsymbol{P}_h^1\boldsymbol{\psi}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega), \quad \boldsymbol{\eta} \in \boldsymbol{S}_h. \tag{6}$$

Therefore we can define $\Pi_h(\boldsymbol{\psi}, \mu) = (\boldsymbol{P}_h^1\boldsymbol{\psi}, \Pi_h^1\mu)$ and condition S1 holds. Again the estimate (4) follows.

The choice of elements (5) uses the velocity and pressure interpolations from the MINI Stokes element for the transverse displacement and the shear stress, respectively, and for the rotations uses a space which contains the space used for the shear stress. Using other Stokes elements introduced in [1] in a similar way we can derive other stable Reissner–Mindlin elements, all with continuous piecewise polynomial approximation of the shear stress. The next simplest approximation would be

$$\boldsymbol{V}_h = \boldsymbol{L}_2^1(\mathcal{T}_h), \quad W_h = L_2^1(\mathcal{T}_h) + B_3(\mathcal{T}_h), \quad \boldsymbol{S}_h = \boldsymbol{L}_1^1(\mathcal{T}_h), \tag{7}$$

shown in Figure 2. This element is the most balanced with respect to approximability of the three components. Instead of (4) we get

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\omega - \omega_h\|_1 + |||\boldsymbol{\zeta} - \boldsymbol{\zeta}_h|||_t \le Ch^2(\|\boldsymbol{\phi}\|_3 + \|\omega\|_3 + \|\boldsymbol{\zeta}\|_2).$$

Note that we used the fact that $\boldsymbol{S}_h \subset \boldsymbol{V}_h$ to obtain (6) and therefore verify condition S1. For the clamped plate problem this inclusion doesn't hold since in that case the functions in $\boldsymbol{V}_h$, but not those in $\boldsymbol{S}_h$, vanish on the boundary. It seems likely that at least for the element choice (7), making use of the additional degrees of freedom for $\boldsymbol{V}_h$ on element edges, it should still be possible to verify S1.

References

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