INTERIOR ESTIMATES FOR A LOW ORDER FINITE ELEMENT METHOD FOR THE REISSNER–MINDLIN PLATE MODEL*

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Abstract. Interior error estimates are obtained for a low order finite element introduced by Arnold and Falk for the Reissner–Mindlin plates. It is proved that the approximation error of the finite element solution in the interior domain is bounded above by two parts: one measures the local approximability of the exact solution by the finite element space and the other the global approximability of the finite element method. As an application, we show that for the soft simply supported plate, the Arnold–Falk element still achieves an almost optimal convergence rate in the energy norm away from the boundary layer, even though optimal order convergence cannot hold globally due to the boundary layer. Numerical results are given which support our conclusion.

Key words. Reissner–Mindlin plate, boundary layer, mixed finite element, interior error estimate

AMS(MOS) subject classifications (1991 revision). 65N30, 73N10

1. Introduction

The Reissner–Mindlin plate model describes deformation of a plate with small to moderate thickness subject to a transverse load. The finite element method for this model has been studied extensively (cf. [6], [9], [12], and references therein) and it has been known for a long time that a direct application of standard finite element methods usually leads to unreasonably small solution, as the plate thickness approaches zero. This is usually called the "locking" phenomenon of the finite element method for the Reissner–Mindlin plate.

Another difficulty in approximating the Reissner–Mindlin plate equations is that the solution possesses boundary layers. The structure of the dependence of the solution on the plate thickness was analyzed in detail by Arnold and Falk [4], [3], and [2].

The purpose of this paper is to obtain interior error estimates for the Arnold–Falk element [1] for the Reissner–Mindlin plate model. This element achieves locking-free first order (optimal) convergence for the hard clamped plate. However, this is not true anymore for the plate under the soft simply supported boundary condition, due to a stronger boundary layer effect. Using interior estimates, we will show that, away from the boundary, (almost) first order convergence can still be obtained.

The techniques used here are similar to those developed by the authors [5] to deal with interior error estimates for the finite element methods for the Stokes equations. As was shown in [8], the Reissner–Mindlin model can be reformulated as a coupled system, consisting of a perturbed Stokes-like system and two Poisson equations using the Helmholtz decomposition theorem. Hence our approach combines interior estimates for finite element methods for the Stokes-like system developed here and the interior estimates for nonconforming methods for the Poisson equation. The latter can be found in [13] and [14].

Interior estimates were first introduced by Nitsche and Schatz [15] for primal finite element methods for second order elliptic equations in 1974 and were applied to the mixed finite element methods for second order linear and quasi-linear elliptic equations in [10]. Recently, L. Gastaldi [11] obtained interior error estimates for the Brezzi–Bathe–Fortin [7] family of mixed elements for the Reissner–Mindlin plate model. Her result is similar in nature to ours, but the technique used there depends heavily on a special feature of the Brezzi–Bathe–Fortin elements for the plate: the *commuting diagram property* relating two finite element spaces in the formulation of the elements. This property was also exploited by Douglas and Milner in their

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paper on interior estimates for mixed finite element methods for second order elliptic equations [10]. Those techniques can not be easily adapted to the problem here.

Though the focus of this paper is on the Arnold–Falk element, our results may apply to other elements based on the same reformulation of the Reissner–Mindlin system—the element by Brezzi and Fortin [8] is such an example.

The organization of the paper is as follows. Section 2 presents the Reissner–Mindlin plate equations and its reformulation using the Helmholtz decomposition of the shear stress. The Arnold–Falk element is introduced in section 3. Section 4 is devoted to the interior duality analysis of the variant of the Stokes system. In section 5 we first obtain the interior estimate of the MINI element (Theorem 5.2) for the Stokes-like system with perturbation and then use it to get the interior estimate of the Arnold–Falk element for the Reissner–Mindlin plate model (Theorem 5.3). As an application, we consider the soft simply supported plate in section 6. We show that away from the boundary layer, (almost) optimal order convergence rate can be obtained. Numerical results are given to confirm the theoretical predictions. Finally, in the appendix, we prove an interior regularity result for the solution of the singularly perturbed Stokes-like system that is used in section 4.

2. Notations and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We will use the usual L^2-based Sobolev spaces $H^s = H^s(\Omega)$, $s \in \mathbb{R}$, with the norm $\|\cdot\|_{s,\Omega}$. Recall that for $s \in \mathbb{R}$, H^{-s} denotes the normed dual of $\mathring{H}^s$, the closure of $C_0^\infty(\Omega)$ in H^s. We write $\|\cdot\|_s$ to denote $\|\cdot\|_{s,\Omega}$ when no confusion can arise. We use the notation $(\cdot\,,\cdot)$ for both the $L^2(\Omega)$-innerproduct and its extension to a pairing of $\mathring{H}^s$ and H^{-s}. If X is any subspace of L^2, then $\hat{X}$ denotes the subspace of elements with zero average values. We use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. This is illustrated in the definitions of the following standard differential operators:

$$\mathbf{grad}\, p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}, \qquad \mathbf{curl}\, p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix}.$$

The letter C denotes a generic constant, not the same in each occurrence, but always independent of the meshsize parameter h, and the plate thickness t.

Let G be an open subset of Ω and s an integer. If $\phi \in H^s(G)$, $\psi \in H^{-s}(G)$, and $\omega \in C_0^\infty(G)$, then

$$|(\omega\phi, \psi)| \le C\|\phi\|_{s,G}\|\psi\|_{-s,G},$$

with the constant C depending only on G, ω, and s. For $\mathbf{\Phi} \in \boldsymbol{H}^s(G)$, $\mathbf{\Psi} \in \boldsymbol{H}^{-s+2}(G)$, $P \in H^s(G)$, and $Q \in H^{-s+2}(G)$, define

$$\mathrm{R}(\omega, \mathbf{\Phi}, \mathbf{\Psi}) = \big(\, \mathcal{C}\mathcal{E}(\omega\mathbf{\Phi}), \mathcal{E}(\mathbf{\Psi})\,\big) - \big(\, \mathcal{C}\mathcal{E}(\mathbf{\Psi}), \mathcal{E}(\omega\mathbf{\Psi})\,\big)$$

and

$$\mathrm{R}'\big(\, \omega, P, Q\,\big) = \big(\, \mathbf{curl}(\omega P), \mathbf{curl}\, Q\,\big) - \big(\, \mathbf{curl}\, P, \mathbf{curl}(\omega Q)\,\big),$$

where $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$ and $\mathcal{C}$ is a fourth order tensor (to be defined below). Then we have

$$|\mathrm{R}\big(\, \omega, \mathbf{\Phi}, \mathbf{\Psi}\,\big) \le C\|\mathbf{\Phi}\|_{t,G}\|\mathbf{\Psi}\|_{-t+1,G} \tag{2.1}$$

and

$$|\mathrm{R}'\big(\, \omega, P, Q\,\big) \le C\|P\|_{t,G}\|Q\|_{-t+1,G}, \tag{2.2}$$

for non-negative integers $t \le s$.

Let Ω denote the region in $\mathbb{R}^2$ occupied by the midsection of the plate, and denote by w and $\boldsymbol{\phi}$ the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. Under the soft simply supported boundary condition, the Reissner–Mindlin plate model determines $(w, \boldsymbol{\phi})$ as the unique solution to the following variational problem:

Find $(w, \boldsymbol{\phi}) \in \mathring{H}^1 \times \boldsymbol{H}^1$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \mathbf{grad}\, w, \boldsymbol{\psi} - \mathbf{grad}\, \mu) = (g, \mu) \quad \text{for all } (\mu, \boldsymbol{\psi}) \in \mathring{H}^1 \times \boldsymbol{H}^1. \tag{2.3}$$

Here g denotes the scaled transverse loading function, t the plate thickness, $\lambda = E\kappa/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, and κ the shear correction factor. The bilinear form a is

$$\begin{aligned} a(\boldsymbol{\phi}, \boldsymbol{\psi}) &= \frac{E}{12(1-\nu^2)} \int_\Omega \left[\left(\frac{\partial \phi_1}{\partial x} + \nu \frac{\partial \phi_2}{\partial y} \right) \frac{\partial \psi_1}{\partial x} + \left(\nu \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) \frac{\partial \psi_2}{\partial y} \right. \\ &\qquad \left. + \frac{1-\nu}{2} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \left(\frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x} \right) \right] \\ &= \int_\Omega \mathcal{C}\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\psi}). \end{aligned}$$

Here, $\mathcal{C}$ is a fourth order tensor defined by the bilinear form a.

Following Brezzi and Fortin [8], equation (2.3) can be reformulated by using the Helmholtz Theorem to decompose the shear strain vector

$$\lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\phi}) = \mathbf{grad}\, r + \mathbf{curl}\, p, \tag{2.4}$$

with $(r, p) \in \mathring{H}^1 \times \hat{H}^1$.

The equation (2.3) now becomes

Find $(r, \boldsymbol{\phi}, p, w) \in \mathring{H}^1 \times \boldsymbol{H}^1 \times \hat{H}^1 \times \mathring{H}^1$ such that

$$(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{H}^1, \tag{2.5}$$

$$\big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}\, p, \boldsymbol{\psi} \big) = \big(\mathbf{grad}\, r, \boldsymbol{\psi} \big) \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}^1, \tag{2.6}$$

$$-(\boldsymbol{\phi}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \hat{H}^1, \tag{2.7}$$

$$(\mathbf{grad}\, w, \mathbf{grad}\, s) = (\boldsymbol{\phi} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s) \quad \text{for all } s \in \mathring{H}^1. \tag{2.8}$$

Generally speaking, the regularity of the solution to the above system is sensitive to the boundary condition. For example, under the hard clamped boundary condition (then, $\boldsymbol{\phi}$ is to be found in the space $\mathring{\boldsymbol{H}}^1$, rather than $\boldsymbol{H}^1$), the estimate

$$\|\boldsymbol{\phi}\|_2 + \|p\|_1 \le C\|g\|_{-1},$$

holds with the constant C independent of the plate thickness t [1]. This guarantees the MINI element to achieve a locking free first order convergence rate for the system (2.6) and (2.7) [1].

But such an inequality, with right-hand side independent of t, is not true anymore for the soft simply supported plate. It is proved by Arnold and Falk [4] that the best estimate can obtain with constant independent of t is

$$\|\boldsymbol{\phi}\|_{3/2} + \|p\|_{1/2} \le C,$$

which is obviously not enough for the MINI element to achieve optimal order convergence. Our goal is to show that in interior regions where the solution is smoother, better convergence rates can be obtained.

3. The Arnold–Falk element

Assume now that the domain Ω is a polygon and let $\mathcal{T}_h$ denote a family of triangulations of Ω, which, for simplicity, we assume to be quasi-uniform. triangulation s of Ω. Denote by $P_k(T)$ the set of polynomials of degree not greater than $k \geq 0$ restricted to T, and consider the following finite element spaces:

$$
\begin{aligned}
&Q_h = \{q \in L^2 : q|_T \in P_0(T), \text{ for all } T \in \mathcal{T}_h\},\\
&P_h = \{p \in H^1 : p|_T \in P_1(T), \text{ for all } T \in \mathcal{T}_h\}, \quad \mathring{P}_h = P_h \cap \mathring{H}^1,\\
&\boldsymbol{V}_h = \{\boldsymbol{\psi} \in \boldsymbol{H}^1 : \boldsymbol{\psi}|_T \in [P_1(T) \oplus B^3(T)]^2, \text{ for all } T \in \mathcal{T}_h\}, \quad \mathring{\boldsymbol{V}}_h = \boldsymbol{V}_h \cap \mathring{\boldsymbol{H}}^1,\\
&W_h = \{w \in L^2 : w|_T \in P_1(T), \text{ for all } T \in \mathcal{T}_h, \text{ and } w \text{ is continuous at midpoints}\\
&\qquad \text{of element edges}\},\\
&\mathring{W}_h = \{w \in W_h : w \text{ vanishes at midpoints of boundary edges}\}.
\end{aligned}
$$

In the above, $B^3(T)$ is the subspace of $\mathring{H}^1(T)$ spanned by the cubic bubble function on T. For $\Omega_0 \subseteq \Omega$, let

$$
\begin{aligned}
P_h(\Omega_0) &= \{p|_{\Omega_0} \mid p \in P_h\}, & \mathring{P}_h(\Omega_0) &= \{p \in P_h \mid \operatorname{supp} p \subseteq \bar{\Omega}_0\},\\
W_h(\Omega_0) &= \{w|_{\Omega_0} \mid w \in W_h\}, & \mathring{W}_h(\Omega_0) &= \{w \in W_h \mid \operatorname{supp} w \subseteq \bar{\Omega}_0\},\\
\boldsymbol{V}_h(\Omega_0) &= \{\boldsymbol{\phi}|_{\Omega_0} \mid \boldsymbol{\phi} \in \boldsymbol{V}_h\}, & \mathring{\boldsymbol{V}}_h(\Omega_0) &= \{\boldsymbol{\phi} \in \boldsymbol{V}_h \mid \operatorname{supp} \boldsymbol{\phi} \subseteq \bar{\Omega}_0\}.
\end{aligned}
$$

Let $P_h^0 : L^2(\Omega) \to Q_h$ be the L^2-projection. Then the finite element of Arnold–Falk for the soft simply supported Reissner–Mindlin plate reads as follows:

Find $(w_h, \boldsymbol{\phi}_h) \in \mathring{W}_h \times \boldsymbol{V}_h$, such that

$$
\big(\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})\big) + \lambda t^{-2}\big(\mathring{\boldsymbol{P}}_h\boldsymbol{\phi}_h - \mathbf{grad}_h\, w_h, \boldsymbol{\psi} - \mathbf{grad}_h\, \mu\big) = \big(g, \mu\big), \tag{3.1}
$$

for all $(\mu, \boldsymbol{\psi}) \in \mathring{W}_h \times \boldsymbol{V}_h$. Here $\mathbf{grad}_h$ is the element-wise gradient operator. Now, a discrete version of the Helmholtz theorem introduced in [1] states that

$$
\boldsymbol{Q}_h = \mathbf{grad}_h\, \mathring{W}_h \oplus \mathbf{curl}\, \hat{P}_h.
$$

Thus the discrete shear vector can be decomposed as

$$
\lambda t^{-2}(\mathbf{grad}_h\, w_h - \mathring{\boldsymbol{P}}_h\boldsymbol{\phi}_h) = \mathbf{grad}_h\, r_h + \mathbf{curl}\, p_h, \quad (r_h, p_h) \in \mathring{W}_h \times \hat{P}_h,
$$

and problem (3.1) can be written equivalently in the form:

Find $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in \mathring{W}_h \times \boldsymbol{V}_h \times \hat{P}_h \times \mathring{W}_h$ such that

$$
\begin{aligned}
(\mathbf{grad}_h\, r_h, \mathbf{grad}_h\, \mu) = (g, \mu) \quad &\text{for all } \mu \in \mathring{W}_h,\\
(\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_h, \boldsymbol{\psi}) = (\mathbf{grad}_h\, r_h, \boldsymbol{\psi}) \quad &\text{for all } \boldsymbol{\psi} \in \boldsymbol{V}_h,\\
-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 \quad &\text{for all } q \in \hat{P}_h,\\
(\mathbf{grad}_h\, w_h, \mathbf{grad}_h\, s) = (\boldsymbol{\phi}_h + \lambda^{-1}t^2\, \mathbf{grad}_h\, r_h, \mathbf{grad}_h\, s,) \quad &\text{for all } s \in \mathring{W}_h.
\end{aligned}
$$

The interior analysis of the Arnold–Falk element will be based on the above system. Hence, this work consists of obtaining interior estimates for the nonconforming element for the Poisson equation and that for the MINI element for system (2.6) and (2.7). Here, we will concentrate on the Stokes-like system and refer to [13] and [14] for interior estimates for the nonconforming method.

Now we state some properties of the finite element spaces. The first one and the last two are standard. The proof of (3.4) can be found in [5], and those for (3.5) and (3.6) can be found in [15]. The inverse inequality for nonconforming elements can be found in [13] and [14].

Let G_0 and G be concentric open disks with $G_0 \Subset G \Subset \Omega$, i.e., $\bar{G}_0 \subset G$ and $\bar{G} \subset \Omega$. Since the triangulation $\mathcal{T}_h$ is assumed to be quasi-uniform, there exists a positive real number h_0 and a set G_h which is a union of triangles and satisfies $G_0 \Subset G_h \Subset G_1$, such that for $h \in \big(0, h_0\big]$, the following properties hold.

Approximation property.

(a) If $\boldsymbol{\phi} \in \boldsymbol{H}^2(G)$, then there exists a $\boldsymbol{\phi}^I \in \boldsymbol{V}_h(\Omega)$ such that

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}^I\|_{1,G} \le Ch \mid \boldsymbol{\phi} \mid_{2,G} . \tag{3.2}$$

(b) If $p \in H^{s+1}(G)$, then there exists a $p^I \in P_h(\Omega)$, such that

$$\|p - p^I\|_{s,G} \le Ch\|p\|_{s+1,G}, \quad s = 0, 1.$$

(c) If $r \in H^2(G)$, then there exists $r^I \in W_h(\Omega)$, such that

$$\|r - r^I\|_{1,G}^h \le Ch\|r\|_{2,G}, \tag{3.3}$$

where

$$(\|u\|_{1,G}^h)^2 = \|u\|_{0,G}^2 + \sum_{T \in G} \| \operatorname{\mathbf{grad}} u\|_{0,T\cap G}^2.$$

Furthermore, if $\boldsymbol{\phi}$, p, and r vanish on $G \setminus \bar{G}_0$, respectively, then $\boldsymbol{\phi}^I$, p^I, and r^I can be chosen to vanish on $\Omega \setminus \bar{G}$.

Superapproximation property. Let $\omega \in C_0^\infty(G_0)$, $\boldsymbol{\phi} \in \boldsymbol{V}_h(\Omega)$, $p \in P_h(\Omega)$, and $r \in W_h(\Omega)$. Then there exist $\boldsymbol{\psi}^I \in \mathring{\boldsymbol{V}}_h(G)$, $q^I \in \mathring{P}_h(G)$, and $s^I \in \mathring{W}_h(G)$, such that

$$\|\omega\boldsymbol{\phi} - \boldsymbol{\psi}^I\|_{1,G} \le Ch\|\boldsymbol{\phi}\|_{1,G}, \tag{3.4}$$

$$\|\omega p - q^I\|_{s,G} \le Ch\|p\|_{s,G}, \quad s = 0, 1, \tag{3.5}$$

$$\|\omega r - s^I\|_{1,G}^h \le Ch\|r\|_{1,G}^h, \tag{3.6}$$

where $C = C(G_0, G, \omega)$.

Inverse inequality. Let t be a nonnegative integer. There is a constant C such that

$$\begin{aligned} \|\boldsymbol{\phi}\|_{1,G_h} &\le Ch^{-1-t}\,\|\boldsymbol{\phi}\|_{-t,G_h}, \\ \|p\|_{1,G_h} &\le Ch^{-1-t}\,\|p\|_{-t,G_h}, \\ \|r\|_{1,G_h}^h &\le Ch^{-1-t}\,\|r\|_{-t,G_h}, \text{ for all } r \in W_h(\Omega). \end{aligned}$$

Stability condition. There exists a constant $\gamma > 0$ such that

$$\inf_{\substack{p \in \mathring{P}_h(G_h) \\ p \neq 0}} \sup_{\substack{\boldsymbol{\phi} \in \mathring{\boldsymbol{V}}_h(G_h) \\ \boldsymbol{\phi} \neq 0}} \frac{(\operatorname{rot} \boldsymbol{\phi}\,, p)_{G_h}}{\|\boldsymbol{\phi}\|_{1,G_h}} \ge \gamma.$$

Before we turn to the next section, we introduce a result on the convergence of the MINI element for the perturbed Stokes-like system.

LEMMA 3.1. *Let G_h be a union of triangles. Then for $\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(G_h)$, $p \in H^1(G_h)$, and $\boldsymbol{F} \in L^2(G_h)$, there exist unique functions $\pi\boldsymbol{\phi} \in \mathring{\boldsymbol{V}}_h(G_h)$ and $\pi p \in P_h(G_h)$ with $\int_{G_h} p = \int_{G_h} \pi p$, such that*

$$\big(C\mathcal{E}(\boldsymbol{\phi} - \pi\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}(p - \pi p), \boldsymbol{\psi} \big) = (\boldsymbol{F}, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h),$$
$$-\big(\boldsymbol{\phi} - \pi\boldsymbol{\phi}, \mathbf{curl}\, q \big) - \lambda^{-1}t^2 \big(\mathbf{curl}(p - \pi p), \mathbf{curl}\, q \big) = 0 \quad \text{for all } q \in P_h(G_h).$$

Moreover,

$$\|\boldsymbol{\phi} - \pi\boldsymbol{\phi}\|_{1,G_h} + \|p - \pi p\|_{0,G_h} + t\| \mathbf{curl}(p - \pi p)\|_{0,G_h}$$
$$\leq C \left[\inf_{q \in P_h(G_h)} (\|p - q\|_{0,G_h} + t\| \mathbf{curl}(p - q)\|_{0,G_h}) + \inf_{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h)} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,G_h} + \|\boldsymbol{F}\|_{0,G_h} \right].$$

Proof. The unique existence of solution $(\pi\boldsymbol{\phi}, \pi p)$ follows from Lax–Milgram lemma. The above estimate can be obtained by following the proof in [1, Theorem 5.5]. ∎

4. Interior Duality Estimates

Let $(w, \boldsymbol{\phi}) \in H^1 \times \boldsymbol{H}^1$ be some solution to the Reissner–Mindlin plate equations and let $(r, p) \in H^1 \times H^1$ be determined by the Helmholtz decomposition (2.4). Regardless of the boundary conditions used to specify the particular solution, $(r, \boldsymbol{\phi}, p, w)$ satisfies

$$(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{H}^1,$$
$$\big(C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}\, p, \boldsymbol{\psi} \big) = \big(\mathbf{grad}\, r, \boldsymbol{\psi} \big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1,$$
$$-(\boldsymbol{\phi}, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}\, p, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{H}^1,$$
$$(\mathbf{grad}\, w, \mathbf{grad}\, s) = (\boldsymbol{\phi} + \lambda^{-1}t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s) \quad \text{for all } s \in \mathring{H}^1.$$

Similarly, regardless of the particular boundary conditions, the finite element solution $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in W_h \times \boldsymbol{V}_h \times P_h \times W_h$ satisfies

$$(\mathbf{grad}_h\, r_h, \mathbf{grad}_h\, \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{W}_h,$$
$$\big(C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}\, p_h, \boldsymbol{\psi} \big) = \big(\mathbf{grad}_h\, r_h, \boldsymbol{\psi} \big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h,$$
$$-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h,$$
$$(\mathbf{grad}_h\, w_h, \mathbf{grad}_h\, s) = (\boldsymbol{\phi}_h + \lambda^{-1}t^2\, \mathbf{grad}_h\, r_h, \mathbf{grad}_h\, s) \quad \text{for all } s \in \mathring{W}_h.$$

Then, together with integration by parts, we obtain

$$(\mathbf{grad}_h(r - r_h), \mathbf{grad}_h\, \mu) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial r}{\partial n} \mu \quad \text{for all } \mu \in \mathring{W}_h, \tag{4.1}$$

$$(C\,\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}(p - p_h), \boldsymbol{\psi}) = (\mathbf{grad}_h(r - r_h), \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.2}$$

$$-(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}(p - p_h), \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h, \tag{4.3}$$

$$(\mathbf{grad}_h(w - w_h), \mathbf{grad}_h\, s) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h + \lambda^{-1}t^2\, \mathbf{grad}_h(r - r_h), \mathbf{grad}_h\, s)$$
$$- \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\phi} \cdot \boldsymbol{n}_T + \lambda^{-1}t^2 \frac{\partial r}{\partial n} - \frac{\partial w}{\partial n}) s \quad \text{for all } s \in \mathring{W}_h. \tag{4.4}$$

Interior estimates for nonconforming methods for the second order elliptic equations have been obtained in [13] and [14]. To study the Stokes-like system, we first consider functions $\boldsymbol{\phi} \in \boldsymbol{H}^1$ and $p \in H^1$ that satisfy

$$(C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p, \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.5}$$

$$-(\boldsymbol{\phi}, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}\, q, \mathbf{curl}\, p) = 0 \quad \text{for all } q \in \mathring{P}_h. \tag{4.6}$$

We have the following result.

THEOREM 4.1. *Assume $\boldsymbol{\phi} \in \boldsymbol{H}^1$ and $p \in H^1$ satisfy* (4.5) *and* (4.6). *Let $G_0 \Subset G \Subset \Omega$ be two concentric disks. Then for any integer $\alpha \geq 0$, the following holds*

$$\|\boldsymbol{\phi}\|_{0,G_0} + \|p\|_{-1,G_0} \leq C(h\|\boldsymbol{\phi}\|_{1,G} + h\|p\|_{0,G} + ht\|\,\mathbf{curl}\,p\|_{0,G} + \|\boldsymbol{\phi}\|_{-\alpha,G} + \|p\|_{-\alpha-1,G}). \tag{4.7}$$

Proof. Choose a disk G_1 such that $G_0 \Subset G_1 \Subset G$ and construct a function $\omega \in C_0^\infty(G_1)$ with $\omega = 1$ on G_0. Then, for any nonnegative integer s,

$$\|\boldsymbol{\phi}\|_{-s,G_0} \leq \|\omega\boldsymbol{\phi}\|_{-s,G} = \sup_{\substack{\boldsymbol{F}\in\mathring{\boldsymbol{H}}^s(G)\\ \boldsymbol{F}\neq 0}} \frac{(\omega\boldsymbol{\phi}, \boldsymbol{F})}{\|\boldsymbol{F}\|_{s,G}}. \tag{4.8}$$

To estimate the right hand side of (4.8), we define $(\boldsymbol{\Phi}, P)$ through (7.1) and (7.2) (in the Appendix) with $K = 0$. Taking $\boldsymbol{\Psi} = \omega\boldsymbol{\phi}$ in (7.1) yields

$$\begin{aligned}
\big(\,\omega\boldsymbol{\phi}, \boldsymbol{F}\,\big) &= \big(\,C\mathcal{E}(\omega\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\Phi})\,\big) - \big(\,\omega\boldsymbol{\phi}, \mathbf{curl}\,P\,\big)\\
&= \big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\omega\boldsymbol{\Phi})\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big) - \mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\phi}) + \big(\,\mathbf{curl}\,\omega, P\boldsymbol{\phi}\,\big)\\
&= \big\{\big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[(\omega\boldsymbol{\Phi})^I]\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big)\big\}\\
&\quad + \big\{\big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[\omega\boldsymbol{\Phi} - (\omega\boldsymbol{\Phi})^I]\,\big) - \mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\phi}) + \big(\,\mathbf{curl}\,\omega, P\boldsymbol{\phi}\,\big)\big\}\\
&=: A_1 + B_1.
\end{aligned} \tag{4.9}$$

Here the superscript I denotes the approximation operator defined in (3.2)–(3.3). Choosing $\boldsymbol{\psi}$ to be $(\omega\boldsymbol{\Phi})^I$ in (4.5) we get

$$\begin{aligned}
A_1 &= \big(\,\mathbf{curl}\,p, (\omega\boldsymbol{\Phi})^I\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big) = \big(\,\mathbf{curl}\,p, \omega\boldsymbol{\Phi}\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big) + \big(\,\mathbf{curl}\,p, (\omega\boldsymbol{\Phi})^I - \omega\boldsymbol{\Phi}\,\big)\\
&= \big\{\big(\,\mathbf{curl}(\omega p), \boldsymbol{\Phi}\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big)\big\} \quad - \big\{\big(\,\mathbf{curl}\,\omega, p\boldsymbol{\Phi}\,\big) - \big(\,\mathbf{curl}\,p, (\omega\boldsymbol{\Phi})^I - \omega\boldsymbol{\Phi}\,\big)\big\} =: A_2 + B_2.
\end{aligned}$$

Taking $Q = \omega p$ in (7.2) ($K = 0$), we obtain

$$\begin{aligned}
A_2 &= -\lambda^{-1}t^2\big(\,\mathbf{curl}(\omega p), \mathbf{curl}\,P\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big)\\
&= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}(\omega P)\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\\
&= \big\{-\lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}[(\omega P)^I]\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big)\big\}\\
&\quad + \big\{\lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}[(\omega P)^I - \omega P]\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\big\} =: A_3 + B_3.
\end{aligned}$$

Substituting $q = (\omega P)^I$ in (4.6), we have

$$A_3 = \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I]\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}(\omega P)\,\big) = \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I - \omega P]\,\big). \tag{4.10}$$

Combining (4.9) through (4.10), we get

$$\big(\,\omega\boldsymbol{\phi}, \boldsymbol{F}\,\big) = B_1 + B_2 + B_3 + A_3. \tag{4.11}$$

Then applying the approximation property, (2.1), (2.2), integration by parts, and the Schwarz inequality, we obtain

$$\begin{aligned}
|\,B_1\,| &\leq C\,[h\|\boldsymbol{\phi}\|_{1,G_1}\|\boldsymbol{\Phi}\|_{2,G_1} + \|\boldsymbol{\phi}\|_{-s-1,G_1}(\|\boldsymbol{\Phi}\|_{s+2,G_1} + \|P\|_{s+1,G_1})]\,,\\
|\,B_2\,| &\leq C\big(\,h\|p\|_{0,G_1}\|\boldsymbol{\Phi}\|_{2,G_1} + \|p\|_{-s-2,G_1}\|\boldsymbol{\Phi}\|_{s+2,G_1}\,\big),\\
|\,B_3\,| &\leq C\big(\,ht^2\|\,\mathbf{curl}\,p\|_{0,G_1}\|P\|_{2,G_1} + t^2\|p\|_{-s-2,G_1}\|P\|_{s+3,G_1}\,\big),\\
|\,A_3\,| &\leq Ch\|\boldsymbol{\phi}\|_{1,G_1}\|P\|_{1,G_1}.
\end{aligned} \tag{4.12}$$

First combining (4.12), (4.11), and (4.8), then applying (7.3) and (7.4) (in the Appendix), we obtain

$$\|\boldsymbol{\phi}\|_{-s,G_0} \le C\big(h\|\boldsymbol{\phi}\|_{1,G} + h\|p\|_{0,G} + \|\boldsymbol{\phi}\|_{-s-1,G} + \|p\|_{-s-2,G} + ht\|\,\mathbf{curl}\, p\|_{0,G}\big). \tag{4.13}$$

To estimate $\|p\|_{-s-1,G_0}$, first find a function $\delta \in C_0^\infty(G_1)$ with $\int_G \delta = 1$. Then,

$$\|p\|_{-s-1,G_0} \le \|\omega p\|_{-s-1,G} = \sup_{\substack{g\in \mathring{H}^{s+1}(G)\\ g\neq 0}} \frac{(\omega p, g)}{\|g\|_{s+1,G}}. \tag{4.14}$$

Note that

$$(\omega p, g) = (\omega p, g - \delta \int_G g) + (\omega p, \delta \int_G g) \tag{4.15}$$

and

$$|\,(\omega p, \delta \int_G g)\,| \le C\|p\|_{-s-1,G}\|g\|_{0,G}.$$

In order to estimate the first term on the right hand side of (4.15), we define $(\boldsymbol{\Phi}, P)$ through (7.1) and (7.2) with $\boldsymbol{F} = 0$, $K = g - \delta\int_G g$. Taking $Q = \omega p$ in (7.2) yields

$$\begin{aligned}
\big(\omega p, g - \delta \int_G g\big) &= -\big(\,\mathbf{curl}(\omega p), \boldsymbol{\Phi}\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}(\omega p), \mathbf{curl}\, P\,\big)\\
&= -\big(\,\mathbf{curl}\, p, \omega\boldsymbol{\Phi}\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}\, p, \mathbf{curl}(\omega P)\,\big) - \big(\,\mathbf{curl}\,\omega, p\boldsymbol{\Phi}\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\\
&= -\big\{\big(\,\mathbf{curl}\, p, (\omega\boldsymbol{\Phi})^I\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\, p, \mathbf{curl}[(\omega P)^I]\,\big)\big\}\\
&\quad + \big\{\big(\,\mathbf{curl}\, p, (\omega\boldsymbol{\Phi})^I - \omega\boldsymbol{\Phi}\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\, p, \mathbf{curl}[(\omega P)^I - \omega P]\,\big)\\
&\quad -\big(\,\mathbf{curl}\,\omega, p\boldsymbol{\Phi}\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\big\} =: C_1 + D_1.
\end{aligned}$$

Applying (4.5) and (4.6) with $\boldsymbol{\psi} = (\omega\boldsymbol{\Phi})^I$ and $q = (\omega P)^I$, respectively, we get

$$\begin{aligned}
C_1 &= -\big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[(\omega\boldsymbol{\Phi})^I]\,\big) + \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I]\,\big)\\
&= -\big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\omega\boldsymbol{\Phi})\,\big) + \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I]\,\big) + \big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[w\boldsymbol{\Phi} - (\omega\boldsymbol{\Phi})^I]\,\big)\\
&= -\big\{\big(\,C\mathcal{E}(\omega\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\Phi})\,\big) - \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I]\,\big)\big\}\\
&\quad + \big\{\big(\,C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[\omega\boldsymbol{\Phi} - (\omega\boldsymbol{\Phi})^I]\,\big) - \mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\phi})\big\} =: C_2 + D_2.
\end{aligned}$$

Taking $\boldsymbol{\Psi} = \omega\boldsymbol{\phi}$ in (7.1) (with $\boldsymbol{F} = 0$), we obtain

$$C_2 = -\big(\omega\boldsymbol{\phi}, \mathbf{curl}\, P\,\big) + \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I]\,\big) = \big(\,\boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I - \omega P]\,\big) + \big(\,\mathbf{curl}\,\omega, P\boldsymbol{\phi}\,\big).$$

So far, we have

$$(\omega p, g - \delta \int g) = D_1 + D_2 + C_2.$$

Then applying (2.1), (2.2), integration by parts, the approximation property, and the Schwarz inequality, we arrive at

$$\begin{aligned}
|\,D_1\,| &\le C\big(\,h\|p\|_{0,G_1}\|\boldsymbol{\Phi}\|_{2,G_1} + ht^2\|\,\mathbf{curl}\, p\|_{0,G}\|P\|_{2,G_1} + \|p\|_{-s-2,G}\|\boldsymbol{\Phi}\|_{s+2,G_1} + t^2\|p\|_{-s-2,G}\|P\|_{s+3,G_1}\,\big),\\
|\,D_2\,| &\le C\big(\,h\|\boldsymbol{\phi}\|_{1,G}\|\boldsymbol{\Phi}\|_{2,G_1} + \|\boldsymbol{\phi}\|_{-s-1,G_1}\|\boldsymbol{\Phi}\|_{s+2,G_1}\,\big),\\
|\,C_2\,| &\le C\big(\,h\|\boldsymbol{\phi}\|_{1,G_1}\|P\|_{1,G_1} + \|\boldsymbol{\phi}\|_{-s-1,G_1}\|P\|_{s+1,G_1}\,\big).
\end{aligned} \tag{4.16}$$

Combining (4.14) through (4.16), together with (7.3) and (7.4), we obtain

$$\|p\|_{-s-1,G_0} \le C\big(\,h\|\boldsymbol{\phi}\|_{1,G} + h\|p\|_{0,G} + \|\boldsymbol{\phi}\|_{-s-1,G} + \|p\|_{-s-2,G} + ht\|\,\mathbf{curl}\, p\|_{0,G}\,\big). \tag{4.17}$$

Finally, (4.7) can be obtained by choosing a sequence of concentric disks $G_0 \Subset G_1 \Subset \ldots \Subset G_t = G$ and iterating (4.13) and (4.17) in s [15, Lemma 4.1]. ∎

5. Interior Error Estimates

In this section we first obtain the interior estimate for the MINI element for the Stokes-like equations with perturbation, then we use it to derive the interior estimate for the Arnold–Falk element. To be specific, Lemma 5.1 gives a bound on functions satisfying a homogeneous discrete Stokes-like system. It is then used with Theorem 4.1 to get the interior estimate for the MINI element for the Stokes-like system (Theorem 5.2). By combining this result with the interior estimate for nonconforming elements ([13], [14, Chapter 2]) we obtain interior estimates for the Arnold–Falk element in Theorem 5.3.

Lemma 5.1. *Suppose* $(\boldsymbol{\phi}_h, p_h) \in \boldsymbol{V}_h \times P_h$ *is such that*

$$(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_h, \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G), \tag{5.1}$$

$$-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h(G). \tag{5.2}$$

Then, for any concentric disk $G_0 \Subset G \Subset \Omega$*,* h *small enough,* α *and* β *any nonnegative integers, we have*

$$\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\mathbf{curl}\, p_h\|_{0,G_0} \le C\left[t^\beta(\|\boldsymbol{\phi}_h\|_{1,G} + \|p_h\|_{0,G} + t\|p_h\|_{1,G}) + \|\boldsymbol{\phi}_h\|_{-\alpha,G} + \|p_h\|_{-\alpha-1,G}\right], \tag{5.3}$$

where $C = C(\alpha, \beta, G_0, G)$*.*

Proof. Let $G_0 \Subset G' \Subset G_1 \Subset G$ with G' a concentric disk and G_h a union of elements with the property $G' \Subset G_h \Subset G_1$. Construct $\omega \in C_0^\infty(G')$ with $\omega \equiv 1$ on G_0. Set $\widetilde{\boldsymbol{\phi}_h} = \omega\boldsymbol{\phi}_h$, $\widetilde{p_h} = \omega p_h$. Then $\widetilde{\boldsymbol{\phi}_h} \in \mathring{\boldsymbol{H}}^1(G_h)$, $\widetilde{p_h} \in H^1(G_h)$. By Lemma 3.1, $\pi\widetilde{\boldsymbol{\phi}_h} \in \mathring{\boldsymbol{V}}_h(G_h)$ and $\pi\widetilde{p_h} \in P_h(G_h)$ can be uniquely determined by the equations

$$\big(\,\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \boldsymbol{\psi}\,\big) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h), \tag{5.4}$$

$$-\big(\,\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{curl}\, q\,\big) - \lambda^{-1} t^2 \big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, q\,\big) = 0 \quad \text{for all } q \in P_h(G_h), \tag{5.5}$$

with $\int_{G_h} \widetilde{p_h} = \int_{G_h} \pi\widetilde{p_h}$. Moreover, we have

$$\begin{aligned}
\|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} &+ \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h})\|_{0,G_h} \\
&\le C\left[\inf_{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}_h} - \boldsymbol{\psi}\|_{1,G_h} + \inf_{q \in P_h(G_h)} (\|\widetilde{p_h} - q\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p_h} - q)\|_{0,G_h})\right] \\
&\le Ch\big(\,\|\boldsymbol{\phi}_h\|_{1,G_h} + \|p_h\|_{0,G_h} + t\|p_h\|_{1,G_h}\,\big),
\end{aligned}$$

where we used the superapproximation property in the last step. By the triangle inequality and the fact that $\omega = 1$ on G_0

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\mathbf{curl}\, p_h\|_{0,G_0} &\le \|\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h}\|_{0,G_h} + t\|\mathbf{curl}\, \widetilde{p_h}\|_{0,G_h} \\
&\le \|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h})\|_{0,G_h} \\
&\quad + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h} + t\|\mathbf{curl}\, \pi\widetilde{p_h}\|_{0,G_h} \\
&\le Ch(\|\boldsymbol{\phi}_h\|_{1,G_h} + \|p_h\|_{0,G_h} + t\|p_h\|_{1,G_h}) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h} + t\|\mathbf{curl}\, \pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned} \tag{5.6}$$

We shall consider $\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}$ first. In (5.4), we take $\boldsymbol{\psi} = \pi\widetilde{\boldsymbol{\phi}_h}$ to obtain

$$\big(\,\mathcal{C}\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) = \big(\,\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) - \big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \pi\widetilde{\boldsymbol{\phi}_h}\,\big). \tag{5.7}$$

We have

$$\begin{aligned}\big(\,\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol\phi_h}),\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h})\,\big) &= \big(\,\mathcal{C}\mathcal{E}(\omega\boldsymbol\phi_h),\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h})\,\big) = \big(\,\mathcal{C}\mathcal{E}(\boldsymbol\phi_h),\mathcal{E}(\omega\pi\widetilde{\boldsymbol\phi_h})\,\big) - \mathrm{R}(\omega,\pi\widetilde{\boldsymbol\phi_h},\boldsymbol\phi)\\ &= \big(\,\mathcal{C}\mathcal{E}(\boldsymbol\phi_h),\mathcal{E}[(\omega\pi\widetilde{\boldsymbol\phi_h})^I]\,\big) + \Big\{\big(\,\mathcal{C}\mathcal{E}(\boldsymbol\phi_h),\mathcal{E}[\omega\pi\widetilde{\boldsymbol\phi_h} - (\omega\pi\widetilde{\boldsymbol\phi_h})^I]\,\big) - \mathrm{R}(\omega,\pi\widetilde{\boldsymbol\phi_h},\boldsymbol\phi)\Big\}\\ &=: \big(\,\mathcal{C}\mathcal{E}(\boldsymbol\phi_h),\mathcal{E}[(\omega\pi\widetilde{\boldsymbol\phi_h})^I]\,\big) + F_1.\end{aligned}$$

Taking $\boldsymbol\psi = (\omega\pi\widetilde{\boldsymbol\phi_h})^I$ in (5.1), we get

$$\begin{aligned}\big(\,\mathcal{C}\mathcal{E}(\boldsymbol\phi_h),\mathcal{E}[(\omega\pi\widetilde{\boldsymbol\phi_h})^I]\,\big) &= \big(\,\mathbf{curl}\,p_h,(\omega\pi\widetilde{\boldsymbol\phi_h})^I\,\big) = \big(\,\mathbf{curl}\,p_h,\omega\pi\widetilde{\boldsymbol\phi_h}\,\big) + \big(\,\mathbf{curl}\,p_h,(\omega\pi\widetilde{\boldsymbol\phi_h})^I - \omega\pi\widetilde{\boldsymbol\phi_h}\,\big)\\ &= \big(\,\mathbf{curl}(\omega p_h),\pi\widetilde{\boldsymbol\phi_h}\,\big) - \Big\{\big(\,\mathbf{curl}\,\omega,p_h\pi\widetilde{\boldsymbol\phi_h}\,\big) - \big(\,\mathbf{curl}\,p_h,(\omega\pi\widetilde{\boldsymbol\phi_h})^I - \omega\pi\widetilde{\boldsymbol\phi_h}\,\big)\Big\}\\ &=: \big(\,\mathbf{curl}\,\widetilde{p_h},\pi\widetilde{\boldsymbol\phi_h}\,\big) + F_2.\end{aligned}\tag{5.8}$$

Combining (5.7)—(5.8) and substituting $q = \pi\widetilde{p_h}$ in (5.5), we obtain

$$\begin{aligned}\big(\,\mathcal{C}\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h}),&\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h})\,\big) = \big(\,\mathbf{curl}\,\pi\widetilde{p_h},\pi\widetilde{\boldsymbol\phi_h}\,\big) + F_1 + F_2\\ &= \big(\,\widetilde{\boldsymbol\phi_h},\mathbf{curl}\,\pi\widetilde{p_h}\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big) + F_1 + F_2\\ &= \big(\,\omega\boldsymbol\phi_h,\mathbf{curl}\,\pi\widetilde{p_h}\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big) + F_1 + F_2\\ &= \big(\,\boldsymbol\phi_h,\mathbf{curl}(\omega\pi\widetilde{p_h})\,\big) - \big(\,\mathbf{curl}\,\omega,\pi\widetilde{p_h}\boldsymbol\phi_h\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big) + F_1 + F_2\\ &= \big\{\big(\,\boldsymbol\phi_h,\mathbf{curl}[(\omega\pi\widetilde{p_h})^I]\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big)\big\}\\ &\qquad + \big\{\big(\,\boldsymbol\phi_h,\mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big) - \big(\,\mathbf{curl}\,\omega,\pi\widetilde{p_h}\boldsymbol\phi_h\,\big)\big\} + F_1 + F_2\\ &=: E_1 + F_3 + F_1 + F_2.\end{aligned}\tag{5.9}$$

Setting $q = (\omega\pi\widetilde{p_h})^I$ in (5.2), we get

$$\begin{aligned}E_1 &= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h,\mathbf{curl}[(\omega\pi\widetilde{p_h})^I]\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big)\\ &= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h,\mathbf{curl}(\omega\pi\widetilde{p_h})\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h,\mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big)\\ &\qquad + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big)\\ &= -\lambda^{-1}t^2(\mathbf{curl}(\omega p_h),\mathbf{curl}\,\pi\widetilde{p_h}) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}),\mathbf{curl}\,\pi\widetilde{p_h}\,\big)\\ &\qquad + \big\{\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h,\mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega,p_h,\pi\widetilde{p_h})\big\}\\ &=: -\lambda^{-1}t^2(\mathbf{curl}\,\pi\widetilde{p_h},\mathbf{curl}\,\pi\widetilde{p_h}) + F_4.\end{aligned}\tag{5.10}$$

In the light of (5.9) and (5.10) we have

$$\big(\,\mathcal{C}\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h}),\mathcal{E}(\pi\widetilde{\boldsymbol\phi_h})\,\big)_{0,G_h} + \lambda^{-1}t^2\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|^2_{0,G_h} = F_1 + F_2 + F_3 + F_4.\tag{5.11}$$

Using the superapproximation property, the Schwartz inequality, integration by parts, (2.1), and (2.2), we obtain

$$\begin{aligned}|\,F_1\,| &\le C\big(\,h\|\boldsymbol\phi_h\|_{1,G_h} + \|\boldsymbol\phi_h\|_{0,G_h}\,\big)\|\pi\widetilde{\boldsymbol\phi_h}\|_{1,G_h},\\ |\,F_2\,| &\le C\big(\,\|p_h\|_{-1,G_h}\| + h\|p_h\|_{0,G_h}\,\big)\|\pi\widetilde{\boldsymbol\phi_h}\|_{1,G_h},\\ |\,F_3\,| &\le C\big(\,h\|\boldsymbol\phi_h\|_{1,G_h} + \|\boldsymbol\phi_h\|_{0,G_h}\,\big)\|\pi\widetilde{p_h}\|_{0,G_h},\\ |\,F_4\,| &\le Ct^2\big(\,h\|\,\mathbf{curl}\,p_h\|_{0,G_h}\|\pi\widetilde{p_h}\|_{1,G_h} + \|p_h\|_{1,G_h}\|\pi\widetilde{p_h}\|_{0,G_h}\,\big).\end{aligned}$$

Applying the above inequalities to (5.11), using the inverse inequalities for $\pi\widetilde{p_h}$, $\boldsymbol\phi_h$, and p_h, and Korn's inequality, we get

$$\begin{aligned}\|\pi\widetilde{\boldsymbol\phi_h}\|^2_{1,G_h} + t^2\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|^2_{0,G_h} &\le C\big(\,h\|\boldsymbol\phi_h\|_{1,G_h} + \|\boldsymbol\phi_h\|_{0,G_h} + h\|p_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\,\big)\|\pi\widetilde{\boldsymbol\phi_h}\|_{1,G_h}\\ &\qquad + C\big(\,h\|\boldsymbol\phi_h\|_{1,G_h} + \|\boldsymbol\phi_h\|_{0,G_h} + t^2\|p_h\|_{1,G_h}\,\big)\|\pi\widetilde{p_h}\|_{0,G_h}\\ &\le C\big(\,\|\boldsymbol\phi_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\,\big)\|\pi\widetilde{\boldsymbol\phi_h}\|_{1,G_h} + C\big(\,\|\boldsymbol\phi_h\|_{0,G_h} + t^2\|p_h\|_{1,G_h}\,\big)\|\pi\widetilde{p_h}\|_{0,G_h}.\end{aligned}\tag{5.12}$$

To proceed, we need to estimate $\|\pi\widetilde{p_h}\|_{0,G_h}$. By the triangle inequality and the fact that $\int_{G_h}\pi\widetilde{p_h} \equiv \int_{G_0}\widetilde{p_h}$, we have

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le \|\pi\widetilde{p_h} - c\int_{G_h}\pi\widetilde{p_h}\|_{0,G_h} + c\|\int_{G_h}\widetilde{p_h}\|_{0,G_h}, \tag{5.13}$$

where $c = 1/\text{meas}(G_h)$. Clearly,

$$\|\int_{G_h}\widetilde{p_h}\|_{0,G_h} = \|\int_{G_h}\omega p_h\|_{0,G_h} \le C\|p_h\|_{-1,G_h}.$$

Since the triangulation $\mathcal{T}_h$ is quasi-uniform on G_h, we have the following stability condition for the MINI element on set G_h,

$$\|\pi\widetilde{p_h} - c\int_{G_h}\pi\widetilde{p_h}\|_{0,G_h} \le C \sup_{\substack{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h(G_h)\\ \boldsymbol{\psi}\neq 0}} \frac{\big(\,\mathbf{curl}\,\pi\widetilde{p_h},\boldsymbol{\psi}\,\big)_{G_h}}{\|\boldsymbol{\psi}\|_{1,G_h}}.$$

Applying (5.4), we obtain

$$\begin{aligned}
(\mathbf{curl}\,\pi\widetilde{p_h},\boldsymbol{\psi}) &= \big(\,\mathbf{curl}\,\widetilde{p_h},\boldsymbol{\psi}\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\\
&= \big(\,\mathbf{curl}(\omega p_h),\boldsymbol{\psi}\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\\
&= \big(\,\mathbf{curl}\,p_h, w\boldsymbol{\psi}\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big) + \big(\,\mathbf{curl}\,\omega, p_h\boldsymbol{\psi}\,\big)\\
&= \Big\{\big(\,\mathbf{curl}\,p_h,(\omega\boldsymbol{\psi})^I\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\Big\} + \big\{\big(\,\mathbf{curl}\,\omega, p_h\boldsymbol{\psi}\,\big) + \big(\,\mathbf{curl}\,p_h,\omega\boldsymbol{\psi} - (\omega\boldsymbol{\psi})^I\,\big)\big\}\\
&=: G_1 + H_1.
\end{aligned}$$

Substituting $(\omega\boldsymbol{\psi})^I$ for $\boldsymbol{\psi}$ in (5.1), we get

$$\begin{aligned}
G_1 &= \big(\,C\mathcal{E}(\boldsymbol{\phi}_h),\mathcal{E}[(\omega\boldsymbol{\psi})^I]\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\\
&= \big(\,C\mathcal{E}(\boldsymbol{\phi}_h),\mathcal{E}(\omega\boldsymbol{\psi})\,\big) + \big(\,C\mathcal{E}(\boldsymbol{\phi}_h),\mathcal{E}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}]\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\\
&= \Big\{\big(\,C\mathcal{E}(\omega\boldsymbol{\phi}_h),\mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big)\Big\} + \big\{\mathrm{R}(\omega,\boldsymbol{\psi},\boldsymbol{\phi}_h) + \big(\,C\mathcal{E}(\boldsymbol{\phi}_h),\mathcal{E}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}]\,\big)\big\}\\
&=: \big(\,C\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big) + H_2.
\end{aligned}$$

So far, we have

$$\big(\,\mathbf{curl}\,\pi\widetilde{p_h},\boldsymbol{\psi}\,\big) = \big(\,C\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi})\,\big) + H_1 + H_2. \tag{5.14}$$

Using the superapproximation property, the Schwartz inequality, (2.1), (2.2), and integration by parts, we have

$$\begin{aligned}
|\,H_1\,| &\le C\big(\,\|p_h\|_{-1,G_h} + h\|p_h\|_{0,G_h}\,\big)\|\boldsymbol{\psi}\|_{1,G_h},\\
|\,H_2\,| &\le C\big(\,\|\boldsymbol{\phi}_h\|_{0,G_h} + h\|\boldsymbol{\phi}_h\|_{1,G_h}\,\big)\|\boldsymbol{\psi}\|_{1,G_h},\\
|\,(\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}),\mathcal{E}(\boldsymbol{\psi}))\,| &\le \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}\|\boldsymbol{\psi}\|_{1,G_h}.
\end{aligned}$$

Combining the above inequalities with (5.13)–(5.14) and using the inverse inequalities, we arrive at

$$\begin{aligned}
\|\pi\widetilde{p_h}\|_{0,G_h} &\le C\big(\,h\|\boldsymbol{\phi}_h\|_{1,G_h} + \|\boldsymbol{\phi}_h\|_{0,G_h} + h\|p_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\big) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}\\
&\le C\big(\,\|\boldsymbol{\phi}_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\,\big) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}.
\end{aligned} \tag{5.15}$$

Substituting (5.15) into (5.12) and using the arithmetic-geometric mean inequality, we have

$$\|\pi\widetilde{\phi_h}\|_{1,G_h} + t\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|_{0,G_h} \le C\big(\,\|\phi_h\|_{0,G_h} + t^2\|p_h\|_{1,G_h} + \|p_h\|_{-1,G_h}\,\big). \tag{5.16}$$

Substituting (5.16) back into (5.15), we get

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le C\big(\,\|\phi_h\|_{0,G_h} + \|p_h\|_{-1,G_h} + t^2\|p_h\|_{1,G_h}\,\big). \tag{5.17}$$

Hence, combining (5.16) and (5.17) with (5.6), we obtain

$$\|\phi_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\,p_h\|_{0,G_0} \le C\left[\|\phi_h\|_{0,G_h} + \|p_h\|_{-1,G_h} + (ht+t^2)\|p_h\|_{1,G_h}\right].$$

Applying Theorem 4.1 with G_0 replaced by G_1 to bound $\|\phi_h\|_{0,G_1}$ and $\|p_h\|_{-1,G_1}$, we get

$$\begin{aligned}\|\phi_h\|_{1,G_0} &+ \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\,p_h\|_{0,G_0} \\ &\le C\left[h\|\phi_h\|_{1,G} + h\|p_h\|_{0,G} + t(t+h)\|p_h\|_{1,G} + \|\phi_h\|_{-\alpha,G} + \|p_h\|_{-\alpha-1,G}\right].\end{aligned} \tag{5.18}$$

In order to prove (5.3), we use the interative method [15, Lemma 5.2]. Let $G_0 \Subset G_1 \Subset \ldots \Subset G_{\beta+1} = G$ be concentric disks and G_h a union of elements with the property $G_\beta \Subset G_h \Subset G$. Introduce the notation

$$S(j) = \|\phi_h\|_{1,G_j} + \|p_h\|_{0,G_j} + t\|\,\mathbf{curl}\,p_h\|_{0,G_j}.$$

Then apply (5.18) to each pair $G_j \Subset G_{j+1}$ (with G_0 and G replaced by G_j and G_{j+1}, respectively) to get

$$S(j) \le C\left[(h+t)S(j+1) + \|\phi_h\|_{-\alpha,G_{j+1}} + \|p_h\|_{-\alpha-1,G_{j+1}}\right] \quad \text{for} \quad j = 0\ldots\beta-1.$$

Iterating the above inequality yields

$$S(0) \le C\left[(h+t)^\beta S(\beta) + \|\phi_h\|_{-\alpha,G_\beta} + \|p_h\|_{-\alpha-1,G_\beta}\right].$$

Let us first assume that $\beta \ge \alpha+1$. If $t \le h$, then using the inverse inequality we get

$$S(0) \le C\big(\,\|\phi_h\|_{-\alpha,G_h} + \|p_h\|_{-\alpha-1,G_h}\,\big).$$

If $t \ge h$, then

$$S(0) \le C\big(\,t^\beta S(\beta) + \|\phi_h\|_{-\alpha,G_\beta} + \|p_h\|_{-\alpha-1,G_\beta}\,\big).$$

Thus the above inequality holds for any h and t.

The restriction $\beta \ge \alpha+1$ be dropped since t can be assumed to be less than one here. Thus (5.3) is proved. ∎

THEOREM 5.2. *Let $\Omega_0 \Subset \Omega_1 \Subset \Omega$, and α and β be two arbitrary nonnegative integers. Suppose that $(\phi,p) \in \boldsymbol{H}^1 \times H^1$ satisfies $\phi|_{\Omega_1} \in \boldsymbol{H}^2(\Omega_1)$ and $p|_{\Omega_1} \in H^2(\Omega_1)$, and $(\phi_h,p_h) \in \boldsymbol{V}_h \times P_h$ are such that*

$$(\mathcal{C}\mathcal{E}(\phi-\phi_h),\mathcal{E}(\psi)) - (\mathbf{curl}(p-p_h),\psi) = (\boldsymbol{F},\psi) \quad \textit{for all } \psi \in \mathring{\boldsymbol{V}}_h, \tag{5.19}$$

$$-(\phi-\phi_h,\mathbf{curl}\,q) - \lambda^{-1}t^2(\mathbf{curl}(p-p_h),\mathbf{curl}\,q) = 0 \quad \textit{for all } q \in \mathring{P}_h. \tag{5.20}$$

for some function $\boldsymbol{F}$ in $\boldsymbol{L}^2$. Then, there is a positive number h_1 such that for $h \in (0,h_1]$,

$$\begin{aligned}\|\phi-\phi_h\|_{1,\Omega_0} &+ \|p-p_h\|_{0,\Omega_0} + t\|\,\mathbf{curl}(p-p_h)\|_{0,\Omega_0} \\ &\le C\left[\|\boldsymbol{F}\|_{0,\Omega_1} + h(\|\phi\|_{2,\Omega_1} + \|p\|_{1,\Omega_1} + t\|p\|_{2,\Omega_1}) + t^\beta(\|\phi-\phi_h\|_{1,\Omega_1}\right. \\ &\qquad \left. + \|p-p_h\|_{0,\Omega_1} + t\|p-p_h\|_{1,\Omega_1}) + \|\phi-\phi_h\|_{-\alpha,\Omega_1} + \|p-p_h\|_{-\alpha-1,\Omega_1}\right],\end{aligned} \tag{5.21}$$

for a constant C depending only on Ω_1, Ω_0, α, and β.

Proof. Let $G_0 \Subset G_0' \Subset G' \Subset G_1 \Subset G$ be concentric disks, G_h a union of elements with $G' \Subset G_h \Subset G_1$, and find $\omega \in C_0^\infty(G')$ with $\omega \equiv 1$ on G_0'. Set $\widetilde{\boldsymbol{\phi}} = \omega\boldsymbol{\phi}$, $\widetilde{p} = \omega p$. Then $\widetilde{\boldsymbol{\phi}} \in \mathring{\boldsymbol{H}}^1(G_h)$, $\widetilde{p} \in H^1(G_h)$. By Lemma 3.1, $\pi\widetilde{\boldsymbol{\phi}} \in \mathring{\boldsymbol{V}}_h(G_h)$, $\pi\widetilde{p} \in P_h(G_h)$ can be defined uniquely by the following equations,

$$(5.22)\qquad \big(\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})\big) - \big(\mathbf{curl}(\widetilde{p} - \pi\widetilde{p}), \boldsymbol{\psi}\big) = (\boldsymbol{F}, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h),$$

$$(5.23)\qquad -\big(\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}, \mathbf{curl}\, q\big) - \lambda^{-1}t^2\big(\mathbf{curl}(\widetilde{p} - \pi\widetilde{p}), \mathbf{curl}\, q\big) = 0 \quad \text{for all } q \in P_h(G_h),$$

with $\int_{G_h} \pi\widetilde{p} = \int_{G_h} \widetilde{p}$. Moreover, we have

$$(5.24)\qquad \begin{aligned} &\|\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_h} + \|\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p} - \pi\widetilde{p})\|_{0,G_h} \\ &\quad\le C\left[\inf_{q\in P_h(G_h)}(\|\widetilde{p} - q\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p} - q)\|_{0,G_h}) + \inf_{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}} - \boldsymbol{\psi}\|_{1,G_h} + \|\boldsymbol{F}\|_{0,G_h}\right] \\ &\quad\le C\big(\|\boldsymbol{\phi}\|_{1,G} + \|p\|_{0,G} + t\|p\|_{1,G} + \|\boldsymbol{F}\|_{0,G_h}\big). \end{aligned}$$

Let us now estimate $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0}$, $\|p - p_h\|_{0,G_0}$, and $t\|\mathbf{curl}(p - p_h)\|_{0,G_0}$. By the triangle inequality, we have

$$(5.25)\qquad \begin{aligned} &\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0} + t\|\mathbf{curl}(p - p_h)\|_{0,G_0} \\ &\quad\le \|\boldsymbol{\phi} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0} + \|p - \pi\widetilde{p}\|_{0,G_0} + t\|\mathbf{curl}(p - \pi\widetilde{p})\|_{0,G_0} \\ &\qquad + \|\pi\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h\|_{1,G_0} + \|\pi\widetilde{p} - p_h\|_{0,G_0} + t\|\mathbf{curl}(\pi\widetilde{p} - p_h)\|_{0,G_0}. \end{aligned}$$

Since $\omega \equiv 1$ on G_0', if we consider (5.22), (5.23) and (5.19), (5.20) for $\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_0')$, $q \in \mathring{P}_h(G_0')$ and subtract corresponding equations, we obtain

$$\begin{aligned} (\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}(p_h - \pi\widetilde{p}), \boldsymbol{\psi}) = 0 &\quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_0'), \\ -(\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}(p_h - \pi\widetilde{p}), \mathbf{curl}\, q) = 0 &\quad \text{for all } q \in \mathring{P}_h(G_0'). \end{aligned}$$

Then we apply Lemma 5.1 to $\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}$ and $p_h - \pi\widetilde{p}$ with G replaced by G_0' to obtain

$$(5.26)\qquad \begin{aligned} &\|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0} + \|p_h - \pi\widetilde{p}\|_{0,G_0} + t\|\mathbf{curl}(p_h - \pi\widetilde{p})\|_{0,G_0} \\ &\quad\le C\left[t^\beta(\|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0'} + \|p_h - \pi\widetilde{p}\|_{0,G_0'} + t\|p_h - \pi\widetilde{p}\|_{1,G_0'}) + \|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{-\alpha,G_0'} + \|p_h - \pi\widetilde{p}\|_{-\alpha-1,G_0'}\right] \\ &\quad\le C\Big[t^\beta(\|\boldsymbol{\phi} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0'} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0'} + \|p - \pi\widetilde{p}\|_{0,G_0'} + \|p - p_h\|_{0,G_0'} \\ &\qquad + t\|p - \pi\widetilde{p}\|_{1,G_0'} + t\|p - p_h\|_{1,G_0'}) + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G_0'} + \|p - p_h\|_{-\alpha-1,G_0'}\Big] \\ &\quad\le C\Big[\|\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_h} + \|\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} + t\|\mathbf{curl}(\widetilde{p} - \pi\widetilde{p})\|_{0,G_h} + t^\beta(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G} \\ &\qquad + \|p - p_h\|_{0,G_0'} + t\|p - p_h\|_{1,G}) + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G} + \|p - p_h\|_{-\alpha-1,G}\Big], \end{aligned}$$

where we use the fact that $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}}$ and $p = \widetilde{p}$ on G_0', $t^\beta \le 1$, and $\int_{G_h}(\widetilde{p} - \pi\widetilde{p}) = 0$ Combining (5.25), (5.26), and (5.24) we obtain

$$\begin{aligned} &\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0} + t\|\mathbf{curl}(p - p_h)\|_{0,G_0} \\ &\quad\le C\big(\|\boldsymbol{F}\|_{0,G} + \|\boldsymbol{\phi}\|_{1,G} + \|p\|_{0,G} + t\|\mathbf{curl}\, p\|_{0,G} + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G} \\ &\qquad + t^\beta\|p - p_h\|_{0,G} + t^{\beta+1}\|p - p_h\|_{1,G} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G} + \|p - p_h\|_{-\alpha-1,G}\big). \end{aligned}$$

Since $[(\boldsymbol{\phi}-\boldsymbol{\psi})-(\boldsymbol{\phi}_h-\boldsymbol{\psi})]$ and $[(p-q)-(p_h-q)]$ also satisfy equations (5.19) and (5.20) for any $\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h$ and $q\in\mathring{P}_h$, we have

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,G_0}+\|p-p_h\|_{0,G_0}+t\|\,\mathbf{curl}(p-p_h)\|_{0,G_0}\le C[\inf_{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h}\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{1,G}+\inf_{q\in\mathring{P}_h}(\|p-q\|_{0,G}+t\|\,\mathbf{curl}(p-q)\|_{0,G})$$
$$+t^{\beta}(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,G}+\|p-p_h\|_{0,G}+t\|p-p_h\|_{1,G})+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,G}+\|p-p_h\|_{-\alpha-1,G}+\|\boldsymbol{F}\|_{0,G}].$$

Applying the approximation properties of the finite element spaces yields a local version of (5.21). Then choosing a finite set of concentric disks $G_0\Subset G_1$ to cover $\bar{\Omega}_0$ and $\bar{\Omega}_1$, respectively, and using a covering argument [15, Theorem 5.1], we obtain the desired result. □

We now state the main result of this paper.

Theorem 5.3. *Let $\Omega_0\Subset\Omega_1\Subset\Omega$, and α and β be two nonnegative integers. Suppose that $(r,\boldsymbol{\phi},p,w)\in H^1\times\boldsymbol{H}^1\times H^1\times H^1$ (the exact solution) satisfies $(r,\boldsymbol{\phi},p,w)|_{\Omega_1}\in H^2(\Omega_1)\times\boldsymbol{H}^2(\Omega_1)\times H^2(\Omega_1)\times H^2(\Omega_1)$, and $(r_h,\boldsymbol{\phi}_h,p_h,w_h)\in W_h\times\boldsymbol{V}_h\times P_h\times W_h$ (the finite element solution) is given so that (4.1)–(4.4) hold. Then there exists a positive number h_1 and a constant C depending only on Ω_0, Ω_1, α, and β, such that for all $h\in(0,h_1]$*

$$\begin{aligned}
\|r-r_h\|^h_{1,\Omega_0}&\le C(h\|r\|_{2,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}),\\
\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_0}+\|p-p_h\|_{0,\Omega_0}+t\|\,\mathbf{curl}(p-p_h)\|_{0,\Omega_0}&\le C[h(\|\boldsymbol{\phi}\|_{2,\Omega_1}+\|p\|_{1,\Omega_1}+t\|p\|_{2,\Omega_1}+\|r\|_{2,\Omega_1})\\
&\quad+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,\Omega_1}+\|p-p_h\|_{-\alpha-1,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}\\
&\quad+t^{\beta}(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_1}+\|p-p_h\|_{0,\Omega_1}+t\|p-p_h\|_{1,\Omega_1})],\\
\|w-w_h\|^h_{1,\Omega_0}&\le C[h(\|\boldsymbol{\phi}\|_{2,\Omega_1}+\|p\|_{1,\Omega_1}+t\|p\|_{2,\Omega_1}+\|r\|_{2,\Omega_1}+\|w\|_{2,\Omega_1})+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,\Omega_1}\\
&\quad+\|p-p_h\|_{-\alpha-1,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}+\|w-w_h\|_{-\alpha,\Omega_1}\\
&\quad+t^{\beta}(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_1}+\|p-p_h\|_{0,\Omega_1}+t\|p-p_h\|_{1,\Omega_1})].
\end{aligned}$$

Proof. Find a subdomain Ω' such that $\Omega_0\Subset\Omega'\Subset\Omega_1$. Applying an interior estimate for nonconforming linear elements [13, Theorem 5.1], [14, Theorem 2.5.2] with Ω_0 replaced by Ω' yields

$$\|r-r_h\|^h_{1,\Omega'}\le C\big(\,h\|r\|_{2,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}\,\big),$$

which also implies the first estimate. Using Theorem 5.2 with $\boldsymbol{F}=\mathbf{grad}_h(r-r_h)$ and Ω_1 replaced by Ω', we obtain

$$\begin{aligned}
&\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_0}+\|p-p_h\|_{0,\Omega_0}+t\|\,\mathbf{curl}(p-p_h)\|_{0,\Omega_0}\\
&\qquad\le C[h(\|\boldsymbol{\phi}\|_{2,\Omega'}+\|p\|_{1,\Omega'}+t\|p\|_{2,\Omega'})+\|\,\mathbf{grad}_h(r-r_h)\|_{0,\Omega'}+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,\Omega'}\\
&\qquad\quad+\|p-p_h\|_{-\alpha-1,\Omega'}+t^{\beta}(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega'}+\|p-p_h\|_{0,\Omega'}+t\|p-p_h\|_{1,\Omega'})]\\
&\qquad\le C[h\big(\,\|\boldsymbol{\phi}\|_{2,\Omega_1}+\|p\|_{1,\Omega_1}+t\|p\|_{2,\Omega_1}+\|r\|_{2,\Omega_1})+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,\Omega_1}\\
&\qquad\quad+\|p-p_h\|_{-\alpha-1,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}+t^{\beta}(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_1}+\|p-p_h\|_{0,\Omega_1}+t\|p-p_h\|_{1,\Omega_1}\,\big)].
\end{aligned}$$

This completes the second estimate. The interior estimate for the transverse displacement can be obtained by applying Theorem 5.2 in [13] to (4.4). We skip the details here. □

6. An Example Application

In this section we consider the Reissner–Mindlin plate under the soft simply supported boundary condition. For a smooth domain Ω and a smooth forcing function g, the exact solution satisfies [4], [14]

$$\|w\|_2 + \|r\|_2 + \|\boldsymbol{\phi}\|_{3/2} + \|p\|_{1/2} \le C,$$

which is not enough for achieving optimal convergence rates even for first order elements.

Here we assume that Ω is a convex polygon. At the moment we can not prove the above estimate for such an Ω. But we have reason to believe that it is true for polygons satisfying some angle conditions. Under this assumption, we have [14]

$$\begin{gathered}
h^{-1/2}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|p - p_h\|_0 + t\|\,\mathbf{curl}(p - p_h)\|_0 \le C_\epsilon h^{1/2} t^{-\epsilon}\|g\|_0, \\
\|w - w_h\|_1^h \le C_\epsilon h t^{-\epsilon}(\|g\|_0 + t^2\|g\|_{1/2}), \\
t\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|p - p_h\|_0 + t^2\|\,\mathbf{curl}(p - p_h)\|_0 \le Ch\|g\|_0, \\
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-1} + \|p - p_h\|_{-2} \le C_\epsilon h t^{-\epsilon}\|g\|_0,
\end{gathered}$$

for an arbitrarily small constant ϵ. The optimal order convergence rate is not achieved for $\boldsymbol{\phi}$ on the whole domain. Applying the above estimates to Theorem 5.3 we obtain:

Theorem 6.1. *Let Ω be a convex polygon and $\Omega_0 \Subset \Omega$ an interior domain. Let g be a smooth function. Assume that $\mathcal{T}_h$ is quasi-uniform. Suppose that $(w, \boldsymbol{\phi})$ solves (2.3) and $(w_h, \boldsymbol{\phi}_h)$ solves (3.1). Then there exists a number $h_1 \ge 0$, such that for all $h \in (0, h_1]$ and an arbitrary small constant ϵ*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_0} \le C_\epsilon\, h t^{-\epsilon}\|g\|_0,$$

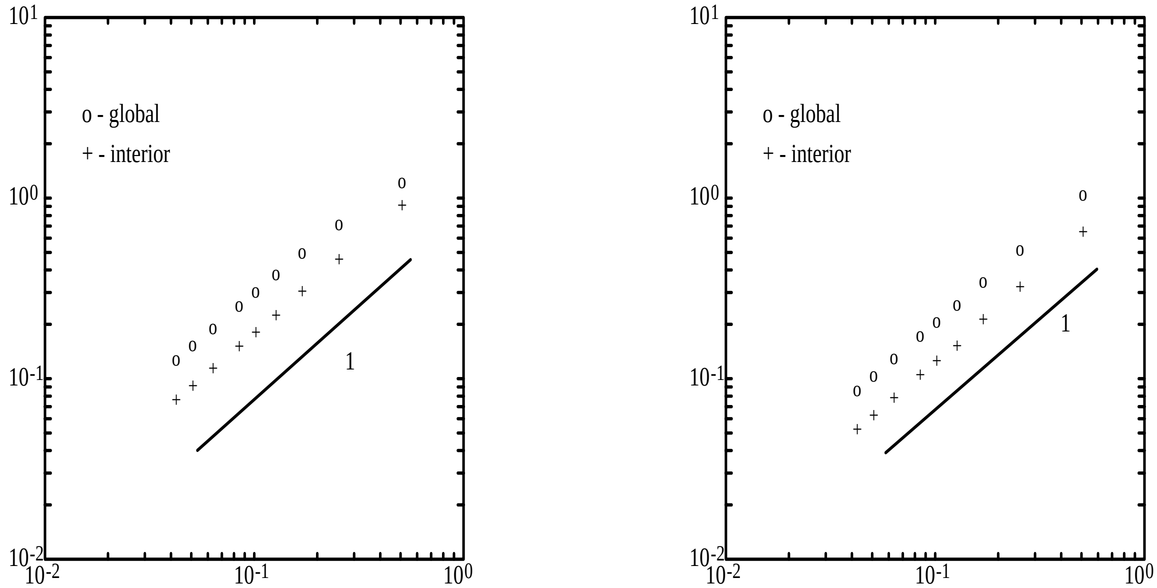
where constant C_ϵ that is independent of t and h.

Now we give the results of computations of the solutions to the Arnold–Falk element for the Reissner–Mindlin plate model. The domain Ω is taken to be the unit square. The exact solution of the semi-infinite ($y > 0$), soft simply supported Reissner–Mindlin plate with forcing function $\cos(x)$ is known [2]. Restrict this solution to Ω and impose the hard clamped boundary condition on the left, upper, and right edges, and soft simply supported boundary conditions on the lower edge. A boundary layer occurs along the lower edge. We take $E = 1$, $\nu = 3/10$, $\kappa = 5/6$, and use a uniform mesh. The interior domain is taken to be the upper half of the unit square. All computations were performed on a Sun SPARCstation 2 using the Modulef package.

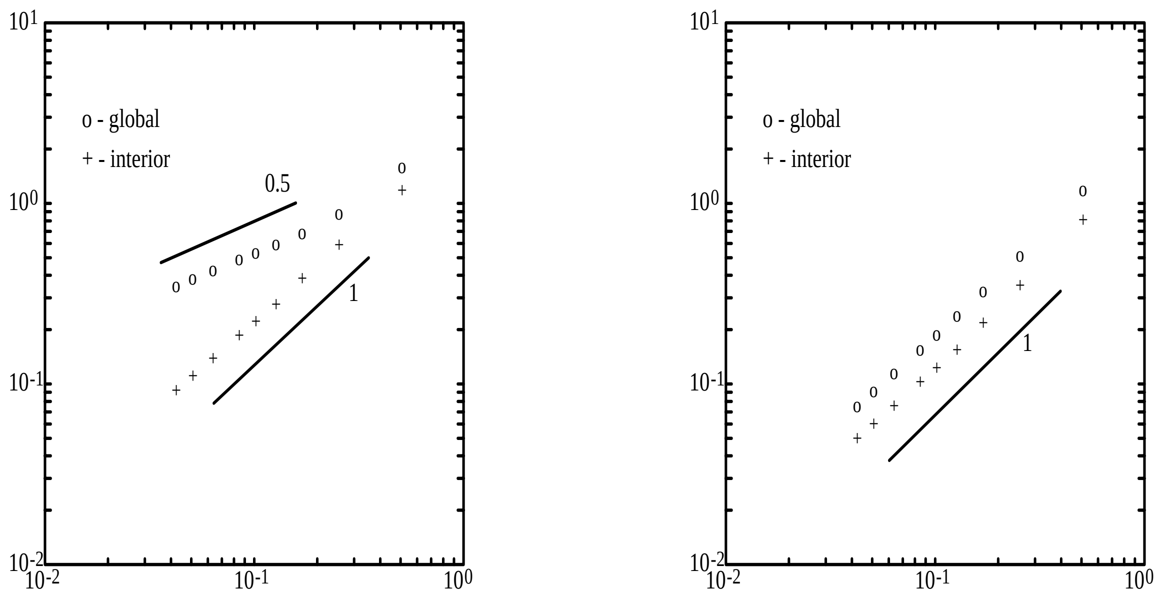
A feature of this test problem is that the exact solution has the property that $\phi_1 \in H^{3/2}(\Omega)$ and $\phi_2 \in H^{5/2}(\Omega)$, i.e., ϕ_1 has a stronger boundary layer than ϕ_2 does (cf. [4]). This is seen in the numerical results.

In the following, the graphs the H^1 norms of the errors on the global domain and the interior domain are plotted as functions of the mesh size h. The values of h are 1/2, 1/4, 1/6, 1/8, 1/10, 1/12, 1/16, 1/20, and 1/24. Both axes have been transformed logarithmically so that the slope of the error curves gives the apparent rate of convergence as h tends to zero. Absolute errors are shown.

Figures 1(a) and 1(b) show the approximation errors in the H^1 norm of components ϕ_1 and ϕ_2 when $t = 1$. Figures 2(a) and 2(b) show the same quantities when $t = 0.0001$. First order optimal convergence is seen as expected when $t = 1$ and there is no difference between the rate on the whole domain and that on the upper half of the unit square. But we only observe $O(h^{1/2})$ order convergence for the global H^1 norm in ϕ_1 when $t = 0.0001$. However, the optimal first order convergence is recovered in the upper half of the unit square.



Figures 1(a), 1(b): Errors for ϕ_1 and ϕ_2 when $t = 1$.



Figures 2(a), 2(b): Errors for ϕ_1 and ϕ_2 when $t = 0.0001$.

7. Appendix: An Interior Regularity Result

In this section we present an interior regularity result for the solution of the singularly perturbed Stokes-like system which was used twice in section 4. It shows that the regularity of the solution in the interior region is not affected by the boundary layer.

The proof basically follows that in [1, Theorem 7.1] for proving the regularity of the solution of the hard-clamped plate and uses the standard approach for analyzing interior regularities for solutions of elliptic equations.

THEOREM 7.1. *Let $\boldsymbol{F} \in \boldsymbol{H}^s(G)$ and $K \in H^{s+1}(G) \cap \hat{L}^2(G)$, where integer $s \geq 0$ and G is a disk. Then there exists a unique solution $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^{s+2}(G) \cap \mathring{\boldsymbol{H}}^1(G) \times H^{s+1}(G) \cap \hat{L}^2(G)$ such that*

$$(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\mathbf{curl}\, P, \boldsymbol{\Psi}) = (\boldsymbol{\Psi}, \boldsymbol{F}) \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G), \tag{7.1}$$

$$-(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - \lambda^{-1} t^2 (\mathbf{curl}\, Q, \mathbf{curl}\, P) = (Q, K) \quad \text{for all } Q \in H^1(G). \tag{7.2}$$

Moreover,

$$\|\boldsymbol{\Phi}\|_{2,G} + \|P\|_{1,G} + t\|P\|_{2,G} + t^2\|P\|_{3,G} \leq C\big(\,\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}\,\big), \tag{7.3}$$

$$\|\boldsymbol{\Phi}\|_{s+2,G_0} + \|P\|_{s+1,G_0} + t\|P\|_{s+2,G_0} + t^2\|P\|_{s+3,G_0} \leq C\big(\,\|\boldsymbol{F}\|_{s,G} + \|K\|_{s+1,G}\,\big), \tag{7.4}$$

for an arbitrary disk $G_0 \Subset G$.

Proof. The inequality

$$\|\boldsymbol{\Phi}\|_{2,G} + \|P\|_{1,G} + t\|P\|_{2,G} \leq C\big(\,\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}\,\big)$$

is proved in [1] for $K = 0$ when $a(\boldsymbol{\Psi}, \boldsymbol{\Phi})$ is simplified into $(\mathbf{grad}\,\boldsymbol{\Psi}, \mathbf{grad}\,\boldsymbol{\Phi})$, i.e., $(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi}))$ is replaced by $(\mathbf{grad}\,\boldsymbol{\Psi}, \mathbf{grad}\,\boldsymbol{\Phi})$ in (7.1). By checking the proof there and using the fact that the bilinear form $\big(\,C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi})\,\big)$ is coercive on the space $\mathring{\boldsymbol{H}}^1$, we can conclude that the same estimate still applies to the current case. We follow the same proof to show that the estimate is still true for $K \neq 0$. At the same time, we will prove that $t^2\|P\|_{3,G}$ is also bounded above by the right hand side of (7.3).

Define $(\boldsymbol{\Phi}^0, P^0) \in \mathring{\boldsymbol{H}}^1(G) \times \hat{\boldsymbol{L}}^2(G)$ as the solution of (7.1) and (7.2) with t set equal to zero:

$$(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi}^0)) - (P^0, \operatorname{rot}\boldsymbol{\Psi}) = (\boldsymbol{\Psi}, \boldsymbol{F}) \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G), \tag{7.5}$$

$$-(\operatorname{rot}\boldsymbol{\Phi}^0, Q) = (Q, K) \quad \text{for all } Q \in L^2(G). \tag{7.6}$$

This is a Stokes–like system which admits a unique solution. Moreover, the standard regularity theory gives [16]

$$\|\boldsymbol{\Phi}^0\|_{2,G} + \|P^0\|_{1,G} \leq C\big(\,\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}\,\big). \tag{7.7}$$

From (7.1), (7.2), (7.5), an (7.6), we get

$$\begin{aligned}
\big(\,C\mathcal{E}(\boldsymbol{\Phi} - \boldsymbol{\Phi}^0), \mathcal{E}(\boldsymbol{\Psi})\,\big) - \big(\,\mathbf{curl}(P - P^0), \boldsymbol{\Psi}\,\big) &= 0 \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G),\\
\big(\,\boldsymbol{\Phi} - \boldsymbol{\Phi}^0, \mathbf{curl}\, Q\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\, P, \mathbf{curl}\, Q\,\big) &= 0 \quad \text{for all } Q \in H^1(G),
\end{aligned}$$

which imply

$$\begin{aligned}
\big(\,C\mathcal{E}(\boldsymbol{\Phi} - \boldsymbol{\Phi}^0), \mathcal{E}(\boldsymbol{\Psi})\,\big) - \big(\,\mathbf{curl}(P - P^0), \boldsymbol{\Psi}\,\big) + \big(\,\boldsymbol{\Phi} - \boldsymbol{\Phi}^0, \mathbf{curl}\, Q\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(P - P^0), \mathbf{curl}\, Q\,\big)&\\
= -\lambda^{-1}t^2\big(\,\mathbf{curl}\, P^0, \mathbf{curl}\, Q\,\big) \quad \text{for all } (\boldsymbol{\Psi}, Q) \in \mathring{\boldsymbol{H}}^1(G) \times H^1(G).&
\end{aligned}$$

Choosing $\boldsymbol{\Psi} = \boldsymbol{\Phi} - \boldsymbol{\Phi}^0$ and $Q = P - P^0$, we obtain

$$\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G}^2 + t^2\|P - P^0\|_{1,G}^2 \leq Ct^2\|P^0\|_{1,G}\|P - P^0\|_{1,G}.$$

It easily follows that

$$\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G} + t\|P - P^0\|_{1,G} \leq Ct\|P^0\|_{1,G} \leq Ct(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}). \tag{7.8}$$

Hence also

$$\|P\|_{1,G} \le C(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}).$$

Applying standard estimates for second-order elliptic problems to (7.1), we further obtain

$$\|\boldsymbol{\Phi}\|_{2,G} \le C(\|P\|_{1,G} + \|\boldsymbol{F}\|_{0,G}) \le C(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}). \tag{7.9}$$

Now from (7.2) and the definition of $\boldsymbol{\Phi}^0$ (i.e., (7.6)) we get

$$\lambda^{-1}t^2(\mathbf{curl}\, P, \mathbf{curl}\, Q) = -(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - (K, Q) = (\boldsymbol{\Phi}^0 - \boldsymbol{\Phi}, \mathbf{curl}\, Q)$$

for all $Q \in H^1(G)$. Thus P is the weak solution of the boundary value problem

$$-\Delta P = \lambda t^{-2} \operatorname{rot}(\boldsymbol{\Phi}^0 - \boldsymbol{\Phi}) \quad \text{in } G, \qquad \frac{\partial P}{\partial n} = 0 \quad \text{on } \partial G,$$

and by standard a priori estimates

$$\|P\|_{2,G} \le Ct^{-2}\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G} \le Ct^{-1}(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,}) \tag{7.10}$$

and

$$\|P\|_{3,G} \le Ct^{-2}\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{2,G} \le Ct^{-2}(\|\boldsymbol{\Phi}\|_{2,G} + \|\boldsymbol{\Phi}^0\|_{2,G}) \le Ct^{-2}(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}), \tag{7.11}$$

where we apply (7.8) in deriving (7.10), and (7.7), (7.9) in deriving (7.11). This completes the proof of (7.3).

In order to prove (7.4), we take a disk G_1 such that $G_0 \Subset G_1 \Subset G$, and a cut-off function $\omega \in C_0^\infty(G_1)$ with $\omega = 1$ on G_0. We will use primes to denote differentian with respect to either x or y, so, for example, P' can be either P_x or P_y. First using integration by parts in (7.1) and (7.2), then differentiating the resulting equations with repect to either x or y, we obtain

$$-\mathbf{div}\, \mathcal{C}\mathcal{E}(\omega\boldsymbol{\Phi}') - \mathbf{curl}(\omega P') = \omega\boldsymbol{F}' - \boldsymbol{J}(\omega, \boldsymbol{\Phi}') - P'\,\mathbf{curl}\,\omega =: \boldsymbol{F}_1,$$
$$-\operatorname{rot}(\omega\boldsymbol{\Phi}') + \lambda^{-1}t^2\,\Delta(\omega P') = \omega K' - \mathbf{curl}\,\omega \cdot \boldsymbol{\Phi}' + \lambda^{-1}t^2\,\Delta\,\omega P' + 2\lambda^{-1}t^2\,\mathbf{grad}\,\omega \cdot \mathbf{grad}\,P' =: K_1,$$

where

$$\boldsymbol{J}(\omega, \boldsymbol{\Phi}') =: \mathbf{div}\, \mathcal{C}\mathcal{E}(\omega\boldsymbol{\Phi}') - \omega\,\mathbf{div}\, \mathcal{C}\mathcal{E}(\boldsymbol{\Phi}').$$

Note that

$$|\boldsymbol{J}(\omega, \boldsymbol{\Phi}')| \le C\|\boldsymbol{\Phi}'\|_{1,G_1}.$$

Clearly,

$$\int_{G_1} K_1 = \int_{G_1} [-\operatorname{rot}(\omega\boldsymbol{\Phi}') + \lambda^{-1}t^2\,\Delta(\omega P')] = -\int_{\partial G_1}\left[\omega\boldsymbol{\Phi}' \cdot \boldsymbol{s} - \lambda^{-1}t^2\frac{\partial(\omega P')}{\partial n}\right] = 0,$$

because both ω and $\mathbf{grad}\,\omega$ vanish on ∂G_1. Moreover, we see that $(\omega\boldsymbol{\Phi}', \omega P')$ satisfies

$$(\mathcal{C}\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\omega\boldsymbol{\Phi}')) - (\mathbf{curl}(\omega P'), \boldsymbol{\Psi}) = (\boldsymbol{\Psi}, \boldsymbol{F}_1) \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G_1),$$
$$-(\omega\boldsymbol{\Phi}', \mathbf{curl}\, Q) - \lambda^{-1}t^2(\mathbf{curl}\, Q, \mathbf{curl}(\omega P')) = (Q, K_1) \quad \text{for all } Q \in H^1(G_1).$$

Thus, (7.3) with $(\boldsymbol{\Phi}, P)$ replaced by $(\omega\boldsymbol{\Phi}', \omega P')$, G replaced by G_1, implies (for $\delta_1 = \int_{G_1} \omega P'$)

$$\begin{aligned}
\|\omega\boldsymbol{\Phi}'\|_{2,G_1} &+ \|\omega P' - \delta_1\|_{1,G_1} + t\|\omega P' - \delta_1\|_{2,G_1} + t^2\|\omega P' - \delta_1\|_{3,G_1} \\
&\le C\big(\|\omega\boldsymbol{F}' - \boldsymbol{J}(\omega, \boldsymbol{\Phi}') - P'\,\mathbf{curl}\,\omega\|_{0,G_1} + \|\omega K' - \mathbf{curl}\,\omega \cdot \boldsymbol{\Phi}' \\
&\qquad + \lambda^{-1}t^2\,\Delta\,\omega P' + 2\lambda^{-1}t^2\,\mathbf{grad}\,\omega \cdot \mathbf{grad}\,P'\|_{1,G_1}\big) \\
&\le C\big(\|\boldsymbol{F}\|_{1,G} + \|K\|_{2,G}\big).
\end{aligned}$$

Since the function ω equals 1 on G_0, the inequality (7.4) is proved for $s = 1$. Next we prove (7.4) for $s = 2$. Using double primes to denote any second order derivative and using the same approach, we can obtain (for the same ω as in before)

$$-\operatorname{\mathbf{div}} \mathcal{C}\mathcal{E}(\omega\mathbf{\Phi}'') - \mathbf{curl}(\omega P'') = \omega \boldsymbol{F}'' - \boldsymbol{J}(\omega, \mathbf{\Phi}'') - P''\,\mathbf{curl}\,\omega =: \boldsymbol{F}_2,$$
$$-\operatorname{rot}(\omega\mathbf{\Phi}'') + \lambda^{-1}t^2\,\Delta(\omega P'') = \omega K'' - \mathbf{curl}\,\omega\cdot\mathbf{\Phi}'' + \lambda^{-1}t^2\,\Delta\,\omega P'' + 2\lambda^{-1}t^2\,\mathbf{grad}\,\omega\cdot\mathbf{grad}\,P'' =: K_2,$$

with $\int_{G_1} K_2 = 0$. Then, the inequality (7.3), with $(\mathbf{\Phi}, P)$ replaced by $(\omega\mathbf{\Phi}'', \omega P'')$ and G replaced by G_1, implies (for $\delta_2 = \int_{G_1}\omega P''$)

$$\begin{aligned}
\|\omega\mathbf{\Phi}''\|_{2,G_1} &+ \|\omega P'' - \delta_2\|_{1,G_1} + t\|\omega P'' - \delta_2\|_{2,G_1} + t^2\|\omega P'' - \delta_2\|_{3,G_1} \\
&\le C\big(\|\omega\boldsymbol{F}'' - \boldsymbol{J}(\omega,\mathbf{\Phi}'') - P''\,\mathbf{curl}\,\omega\|_{0,G_1} \\
&\qquad + \|\omega K'' - \mathbf{curl}\,\omega\cdot\mathbf{\Phi}'' + \lambda^{-1}t^2\,\Delta\,\omega P'' + 2\lambda^{-1}t^2\,\mathbf{grad}\,\omega\cdot\mathbf{grad}\,P''\|_{1,G_1}\big) \\
&\le \big(\|\boldsymbol{F}\|_{2,G_1} + \|K\|_{3,G_1} + \|\mathbf{\Phi}\|_{3,G_1} + \|P\|_{2,G_1} + t\|P\|_{3,G_1} + t^2\|P\|_{4,G_1}\big) \\
&\le C\big(\|\boldsymbol{F}\|_{2,G} + \|K\|_{3,G}\big),
\end{aligned}$$

where in the last step we use (7.4) with $s = 1$ and G_0 replaced by G_1. Since $\omega = 1$ on G_0, so the inequality (7.4) is proved for $s = 2$. The same arguments, together with an induction on s can be used to prove (7.4) for $s \ge 3$. $\square$

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