THE EFFECT OF THE TEST FUNCTIONS ON THE CONVERGENCE OF SPLINE PROJECTION METHODS FOR SINGULAR INTEGRAL EQUATIONS

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Summary

We investigate the asymptotic convergence properties of a variety of methods for the numerical solution of the system of singular integral equations arising from the traction problem of plane elasticity. Various sorts of Galerkin methods and collocation methods are considered, all of which determine a spline approximation via pairing with certain test functions; the test functions may be splines of the same degree as the trial functions (ordinary Galerkin methods), splines of different degree (Petrov-Galerkin methods), delta functions (collocation), or trigonometric polynomials (spline-trig methods). The choice of test functions is shown to have a significant influence on the convergence properties.

Introduction

We consider here a variety of numerical methods which can be used to solve operator equations, in particular singular integral equations. The methods are (ordinary) spline Galerkin methods, spline Petrov-Galerkin methods, spline collocation methods, and the spline-trig methods. All these methods are variational methods [2] employing splines as trial functions. They differ only in the choice of test functions. The major point of this note is to clarify the effect of the choice of test functions on the asymptotic convergence properties of the methods. The basic conclusion can be summarized as follows. The methods employing smoother test functions have better convergence properties than those with rougher test functions. However the improvement cannot be measured in L^2 or in the Sobolev spaces H^s for $|s|$ small. Rather the advantage can be measured in the very weak norms H^s with $s << 0$.

This conclusion could be justified for quite general operator equations. However the results have a particular significance for the equations arising from boundary integral methods for elliptic boundary value problems, because as explained below, error estimates in such weak Sobolev norms imply error estimates for the solution to the boundary value problem away from the boundary in any norm. Hence in the next section I recall the classical boundary integral formulation of the exterior traction problem in plane elasticity, and shall employ the resulting system of singular integral equations as a model problem throughout the paper.

Most of the results discussed here derive from work of author, much in collaboration with others. In particular several of the results stated here were established in collaboration with W. Wendland [3,4,5], and others were motivated by that collaboration. Specific reference will be given as the results are stated.

Finally let me emphasize that the scope of this note is quite limited and presents only one aspect of a practical comparison of the various methods considered. Indeed, I compare only the convergence properties of the various methods, but not their ease of implementation. Moreover I do not consider at all the important question of numerical quadrature. The comparisons are based only on asymptotic convergence properties that can be rigorously proved or disproved rather than on numerical experiments. Finally I consider singular integral equations posed over a smooth simple closed curve. Although this setting can be generalized in various directions, most of the methods of analysis do not generalize easily to equations posed instead on surfaces, nor do I address the significant difficulties which arise if the curve has corners or endpoints.

1. The Model Problem

Let Γ be a smooth simple closed curve in the plane and Ω its exterior domain. Consider the traction problem for a homogeneous isotropic elastic body in equilibrium:

$$\begin{aligned} \underset{\sim\sim}{Lu} &= 0 \quad \text{on } \Omega, \\ \underset{\sim\sim}{tu} &= \underset{\sim}{\psi} \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

Here

$$\begin{aligned} \underset{\sim\sim}{Lu} &= \underset{\sim}{\text{div}}[2\mu \underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \lambda(\text{div } \underset{\sim}{u})\underset{\approx}{\delta}] \\ &= \mu \underset{\sim}{\Delta} \underset{\sim}{u} + (\lambda+\mu)\underset{\sim}{\text{grad}} \text{ div } \underset{\sim}{u}. \end{aligned}$$

$$\underset{\sim\sim}{tu} = 2\mu \underset{\approx}{\varepsilon}(\underset{\sim}{u})\underset{\sim}{n} + \lambda(\text{div } \underset{\sim}{u})\underset{\sim}{n}.$$

We use the notation $\underset{\approx}{\varepsilon}(\underset{\sim}{u})$ for the strain tensor, $\underset{\sim}{n}$ for the exterior normal, $\underset{\approx}{\delta}$ for the identity tensor, μ and λ for the (positive) Lamé constants. As is well known the problem (1) can be reduced to a system of singular integral equations. First, it suffices to find the displacement on Γ, since by Somigliana's identity

$$\underset{\sim}{u}(\underset{\sim}{z}) = \int_\Gamma [\underset{\approx}{T}(\underset{\sim}{y},\underset{\sim}{z})\underset{\sim}{u}(\underset{\sim}{y}) - \underset{\approx}{E}(\underset{\sim}{y},\underset{\sim}{z})\underset{\sim}{\psi}(\underset{\sim}{y})]d\sigma_{\underset{\sim}{y}}, \qquad \underset{\sim}{z} \in \Omega, \tag{2}$$

where $\underset{\approx}{E}$ is the fundamental solution and the rows of $\underset{\approx}{T}$ the associated tractions (with respect to $\underset{\sim}{y}$). We remark that $\underset{\approx}{T}$ and $\underset{\approx}{E}$ are smooth functions of $\underset{\sim}{y}$ for each $\underset{\sim}{z} \in \Omega$. Letting $\underset{\sim}{z}$ tend to Γ in (2) and using a jump relation we get a vector singular integral equation for the unknown boundary displacement,

$$\underset{\sim}{u}(\underset{\sim}{z}) - 2\int_\Gamma \underset{\approx}{T}(\underset{\sim}{y},\underset{\sim}{z})\, \underset{\sim}{u}(\underset{\sim}{y})d\sigma_{\underset{\sim}{y}} = \underset{\sim}{F}(\underset{\sim}{z}), \qquad \underset{\sim}{z} \in \Gamma, \tag{3}$$

with $\underset{\sim}{F}(\underset{\sim}{z}) = -2\int_\Gamma \underset{\approx}{E}(\underset{\sim}{y},\underset{\sim}{z})\underset{\sim}{\psi}(\underset{\sim}{y})d\sigma_{\underset{\sim}{y}}$. The kernel $\underset{\approx}{T}$ has the form

$$-2\underset{\approx}{T}(\underset{\sim}{y},\underset{\sim}{z}) = \frac{1}{\pi}\underset{\approx}{M}\frac{\partial}{\partial\sigma_{\underset{\sim}{y}}}\log|\underset{\sim}{y}-\underset{\sim}{z}| + \underset{\approx}{K}_1(\underset{\sim}{y},\underset{\sim}{z})$$

where

$$\underset{\approx}{M} = \frac{\mu}{\lambda+2\mu}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\underset{\approx}{K}_1$ is bounded. It is well-known that (3) admits a unique solution. For details on this formulation see [6,7,8].

For computational purposes it is useful to parametrize Γ by a smooth 1-periodic function $\underset{\sim}{x} : \mathbb{R} \to \mathbb{R}^2$. We assume that $\underset{\sim}{x}|_{[0,1)}$ is one-to-one and onto Γ and that $\underset{\sim}{x}'$ never vanishes. Then, in terms of $\underset{\sim}{\phi}(s) = \underset{\sim}{u}(\underset{\sim}{x}(s))$, we may rewrite (3) as

$$\underset{\sim}{\phi}(s) + \frac{1}{\pi} \underset{\approx}{M} \int_0^1 \frac{\partial}{\partial t} \log|\underset{\sim}{x}(t) - \underset{\sim}{x}(s)| \underset{\sim}{\phi}(t)\, dt \quad (4)$$

$$+ \int_0^1 \underset{\approx}{K}_1(\underset{\sim}{x}(t), \underset{\sim}{x}(s)) \underset{\sim}{\phi}(t)\, |\underset{\sim}{x}'(t)| dt = \underset{\sim}{F}(\underset{\sim}{x}(s)), \qquad s \in \mathbb{R}.$$

Now $\frac{\partial}{\partial t} \log|\underset{\sim}{x}(t) - \underset{\sim}{x}(s)|$ differs from $\pi \cot \pi(t-s)$ by a smooth function so (4) can be written

$$A\underset{\sim}{\phi}(s) := \underset{\sim}{\phi}(s) + \underset{\approx}{M} \int_0^1 \cot \pi(t-s)\, \underset{\sim}{\phi}(t) dt$$

$$+ \int_0^1 \underset{\approx}{K}(t,s)\, \underset{\sim}{\phi}(t) dt = \underset{\sim}{f}(s), \qquad s \in \mathbb{R}, \qquad (5)$$

where $\underset{\approx}{K}$ is a bounded kernel and $\underset{\sim}{f}(s) = \underset{\sim}{F}(\underset{\sim}{x}(s))$. We consider here the numerical solution of equations of the form of (5). We shall bear in mind, however, that once we have approximated $\underset{\sim}{\phi}$ we evaluate the displacement or stress in Ω by integrating $\underset{\sim}{\phi}$ against a smooth kernel on Γ. This remark has consequences for the choice of numerical method.

2. Mapping Properties in the Sobolev Spaces

Let ψ and χ be 1-periodic real-valued distributions on $\mathbb{R}$. For $s \in \mathbb{R}$ let

$$\langle \psi, \chi \rangle_s = \sum_{k \in \mathbb{Z}} \hat{\psi}(k)\, \overline{\hat{\chi}(k)}\, \underline{k}^{2s}, \qquad (6)$$

$$\|\psi\|_s = \sqrt{\langle \psi, \psi \rangle_s}, \qquad (7)$$

where $\sum_k \hat{\psi}(k) e^{2\pi i k x}$ is the Fourier series of ψ and $\underline{k} = \max(2\pi|k|, 1)$. The norm (6) is the norm in H^s, the Sobolev space of order s, which consists of the distributions for which the norm is finite. The inner product (7) extends to a pairing on $H^t \times H^{2s-t}$ for all $t \in \mathbb{R}$. We use an undertilde to denote the vector-valued analogues. E.g., $\underset{\sim}{H}^s = H^s \times H^s$ consists of $\mathbb{R}^2$-valued distributions $\underset{\sim}{\psi}$ for which $\|\underset{\sim}{\psi}\|_s^2 := \|\psi_1\|_s^2 + \|\psi_2\|_s^2 < \infty$.

Let

$$A_0 \underset{\sim}{\psi}(s) = \underset{\sim}{\psi}(s) + \underset{\approx}{M} \int_0^1 \cot \pi(t-s)\, \underset{\sim}{\psi}(t) dt$$

denote the principle part of A. Then

$$(A_0 \underset{\sim}{\psi})\hat{}\,(k) = \begin{bmatrix} 1 & ia \\ -ia & 1 \end{bmatrix} \hat{\underset{\sim}{\psi}}(k), \qquad a = \frac{\mu}{\lambda + 2\mu} < \frac{1}{2}.$$

Hence

$$\langle A_0 \underset{\sim}{\psi}, \underset{\sim}{\psi} \rangle_s \geq (1-a)\, \|\underset{\sim}{\psi}\|_s^2. \qquad (8)$$

From this coercivity estimate, the compactness of $A - A_0$ on $\underset{\sim}{H}^s$, (which follows from the decomposition of $\underset{\approx}{K}$ as a weakly singular convolution kernel and a smooth kernel), and the invertibility of A on $\underset{\sim}{H}^0 = \underset{\sim}{L}^2$, it follows that A maps $\underset{\sim}{H}^s$ isomorphically onto itself.

Now suppose that we find an approximation $\bar{\underset{\sim}{\phi}}$ of $\underset{\sim}{\phi}$ which is accurate in the Sobolev norm of order $s < 0$. If $\underset{\approx}{G}$ is a smooth periodic function then

$$\left| \int_0^1 \underset{\approx}{G} \underset{\sim}{\phi} - \int_0^1 \underset{\approx}{G} \bar{\underset{\sim}{\phi}} \right| \leq \|\underset{\approx}{G}\|_{-s}\, \|\underset{\sim}{\phi} - \bar{\underset{\sim}{\phi}}\|_s.$$

In particular error estimates for $\underset{\sim}{\phi}$ even in a very weak norm ($s \ll 0$) provide pointwise error estimates for the stresses and displacements in Ω of the same order, and hence are of direct practical interest. We shall see that approximation in negative norms is closely tied to the choice of *test functions* in the numerical methods.

3. Spline Petrov-Galerkin Methods

Let $\Delta = \{x_m\}_{m \in \mathbb{Z}}$ be a periodic mesh on $\mathbb{R}$ ($x_{m+N} = x_m + 1$ for all m), $d \in \mathbb{N}$. We denote by $S_d(\Delta)$ the N dimensional space consisting of periodic piecewise polynomials of degree d which admit $d-1$ continuous derivatives, and set $\underset{\sim}{S}_d(\Delta) = S_d(\Delta) \times S_d(\Delta)$. The choice of one of these spaces as trial space and one as test space yields a spline Petrov-Galerkin method:

Find $\underset{\sim}{\phi}_\Delta \in \underset{\sim}{S}_d(\Delta)$ such that

$$\int_0^1 A\underset{\sim}{\phi}_\Delta \cdot \underset{\sim}{\psi} = \int_0^1 \underset{\sim}{f} \cdot \underset{\sim}{\psi}, \qquad \underset{\sim}{\psi} \in \underset{\sim}{S}_e(\Delta). \qquad (9)$$

If $d = e$ this is an ordinary Galerkin method (or Bubnov-Galerkin method). We require now that d and e have the same parity (however see § 6). Even for the trivial case A = identity this condition is necessary for nonsingularity of the associated matrix.

The key to the analysis of this method is the following lemma.

<u>Lemma 1</u>: Given $d, e \in \mathbb{N}$ of the same parity, set $j = (d-e)/2$. Then for any $f \in \cup\{H^s |\ s > -e-1/2\}$,

$$\int f\psi = 0 \qquad \text{for all } \psi \in S_e(\Delta)$$

if and only if

$$\langle f, \chi \rangle_j = 0 \qquad \text{for all } \chi \in S_d(\Delta).$$

<u>Proof</u>. Define $D\psi = \psi' + \hat{\psi}(0)$. Note that D maps H^{s+1} isometrically onto H^s for all $s \in \mathbb{R}$ and maps $S_{s+1}(\Delta)$ onto $S_s(\Delta)$ for $s \in \mathbb{N}$. In particular $D^{2j} S_d(\Delta) = S_e(\Delta)$. Since

$$\int f\psi = \sum \hat{f}(k)\, \overline{\hat{\psi}(k)}$$

$$= \hat{f}(0)\, \overline{\hat{\psi}(0)} + (-1)^j \sum_{k \neq 0} \hat{f}(k) \overline{(2\pi i k)^{-2j}\, \hat{\psi}(k)}\, |2\pi k|^{2j}$$

$$= [1 - (-1)^j]\, \hat{f}(0)\, \overline{\hat{\psi}(0)} + (-1)^j \langle f, D^{-2j}\psi \rangle_j,$$

the lemma follows.

<u>Corollary 2</u>: The solution $\underset{\sim}{\phi}_\Delta \in \underset{\sim}{S}_d(\Delta)$ of (9) is characterized by the equations

$$\langle A\underset{\sim}{\phi}_\Delta, \underset{\sim}{\chi} \rangle_j = \langle A\underset{\sim}{\phi}, \underset{\sim}{\chi} \rangle_j, \qquad \underset{\sim}{\chi} \in \underset{\sim}{S}_d(\Delta),$$

where $j = (d-e)/2$.

This characterization of the method allows us to apply the well-developed theory of Galerkin methods. From the invertibility of A and the fact that it is a compact perturbation of an operator which is coercives in the sense of (8), error estimates follow easily. Here we only state the results. For detailed proofs and references in a similar case, see [3].

Theorem 3: For $h_\Delta = \max(x_m - x_{m-1})$ sufficiently small there exists a unique solution to (9). Moreover if $\underset{\sim}{f} \in \underset{\sim}{H}^t$ for some $t \in [(d-e)/2, d+1]$ and $s \in [-e-1, (d-e)/2]$, then

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_\Delta\|_s \le C h_\Delta^{t-s} \|\underset{\sim}{f}\|_t. \qquad (10)$$

The estimate (10) also holds for $s \in [(d-e)/2, d+1/2)$, $s \le t$, with the constant C depending however on the mesh ratio $h_\Delta/\min(x_m - x_{m-1})$.

To better appreciate this result let us suppose that $\underset{\sim}{f}$ is smooth and that the meshes under consideration form a quasiuniform family. Then we have the optimal order estimate

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_\Delta\|_s \le C h^{d+1-s} \|\underset{\sim}{f}\|_{d+1} \qquad (11)$$

for $-e-1 \le s < d+1/2$. The upper limit $s < d + 1/2$ comes from the trial functions and is perfectly natural. The spline space $S_d(\Delta) \subset H^s$ only for such s. The lower limit, $-e-1$, is determined by the test functions. We remark that this limit is real, not an artifact of the method of proof. Indeed, it can be shown that the rate of convergence $O(h^{d+e+2})$ obtained in $\underset{\sim}{H}^{-e-1}$ is the best possible in any Sobolev space. In particular we see that to achieve higher order estimates in a fixed Sobolev space we should increase the degree of the trial functions. However to obtain optimal order estimates in weaker Sobolev spaces we should increase the degree of the test functions.

4. Very Rough Test Functions: The Collocation Method

The collocation method determines $\underset{\sim}{\phi}_\Delta \in \underset{\sim}{S}_d(\Delta)$, d odd, by the equations

$$A\underset{\sim}{\phi}_\Delta(x_m) = \underset{\sim}{f}(x_m) \qquad \text{for all } m. \qquad (12)$$

Defining $S_{-1}(\Delta)$ to be the span of the periodic Dirac distributions δ_{x_m}, we may formally interpret (12) as a Petrov-Galerkin method of the form (9) with $e = -1$. Theorem 3 holds also in this case. The proof outlined in Section 3 needs only to be modified slightly to encompass this case. The modification results from the fact that $D^{d+1}S_d(\Delta)$ almost, but not quite, coincides with $S_{-1}(\Delta)$. The intersection is in fact $N - 1$ dimensional:

$$D^{d+1}S_d(\Delta) \cap S_{-1}(\Delta) = \{\sum_{m=1}^{N} \underset{\sim}{a}_m \delta_{x_m} : \sum_{m=1}^{N} a_m = 0\},$$

while the constant functions are in $D^{d+1}S_d(\Delta)$ but not $S_{-1}(\Delta)$. This difference results in the following variant of Lemma 1, in which we use the notation $J_\Delta \underset{\sim}{f} = 1/2 \sum_{m=1}^{N} (x_{m+1} - x_{m-1})\, \underset{\sim}{f}(x_m)$ for the trapezoidal approximation of $J\underset{\sim}{f} := \hat{\underset{\sim}{f}}(0) = \int_0^1 f$.

Lemma 4 [3]: Let $d \in \mathbb{N}$ be odd, $j = (d+1)/2$. Then for any $f \in \cup\{H^s : s > 1/2\}$,

$$f(x_m) = 0 \qquad \text{for all } m$$

if and only if

$$\langle f + (J_\Delta - J)f, \chi\rangle_j = 0 \qquad \text{for all } \chi \in S_d(\Delta).$$

Since the additional term $J_\Delta - J$ is small, the proof of Theorem 3 in the case $e = -1$ can be carried out. See [3] for details.

An important conclusion is that the collocation method converges with order $d + 1$ at best while an ordinary Galerkin method employing splines of the same degree achieves twice the order, and a Petrov-Galerkin with smoother test function achieves even higher order. Of course the collocation method is less expensive to implement. A comparison taking into account the rates of convergence and the computational complexity is made in [4].

5. Very Smooth Test Functions: The Spline-Trig Method.

The previous results suggest the use of splines of very high degree as test functions. Since there does not appear to be any natural upper limit on the degree, it is reasonable to consider the use of a limiting space "$\lim_{e\to\infty} S_e(\Delta)$". Let us define this space to be

$$\{\psi \in L^2 \mid \exists s_e \in S_e(\Delta),\ e \in \mathbb{N} \text{ odd, such that } s_e \to f \text{ in } L^2 \text{ as } e \to \infty\}.$$

In case Δ is a uniform mesh of size $1/N$, N odd, we can identify this space. It is precisely the space, T_N, of trigonometric polynomials, spanned by 1, $\sin 2\pi x$, $\cos 2\pi x$, $\sin 4\pi x, \ldots, \cos(N-1)\pi x$. This follows from the characterization of the spline spaces in terms of Fourier series; see (14) below and [1].

Thus we are lead to consider the spline-trig method: Find $\underset{\sim}{\phi}_\Delta \in \underset{\sim}{S}_d(\Delta)$ such that

$$\int A\underset{\sim}{\phi}_\Delta \cdot \underset{\sim}{\psi} = \int \underset{\sim}{f} \cdot \underset{\sim}{\psi} \qquad \text{for all } \underset{\sim}{\psi} \in \underset{\sim}{T}_N.$$

This method was first formulated by the author in [1], where an integral equation of the first kind with logarithmic kernel on a plane curve was considered in connection with the Dirichlet problem for Laplace's equation. It was shown that the spline-trig method provides optimal order estimates in <u>all</u> the spaces H^s, $s < 0$. As a consequence the resulting approximation to the solution of Laplace's equation was shown to converge *very* rapidly away from the boundary, faster than any power of h for a C^∞ boundary and like $e^{-c/h}$ for an analytic boundary. It is remarkable that we can achieve infinite order accuracy pointwise even for a method based on piecewise linear (or constant) trial functions. The analysis can be adapted to a wide class of operator equations, among them the singular integral equation considered here. We sketch here some of the main ideas.

The operator A splits as $B + C$ where B is an isomorphism $\underset{\sim}{H}^s \to \underset{\sim}{H}^s$ which is of convolution type, i.e.,

$$(B\underset{\sim}{\psi})\hat{}(k) = \underset{\approx}{m}(k)\, \hat{\underset{\sim}{\psi}}(k),$$

with $\underset{\approx}{m}(k)$ a nonsingular matrix, and C maps $\underset{\sim}{H}^s$ compactly into itself. Here B is the sum of A_0 plus the integral operator arising from the convolutional part of $\underset{\approx}{K}$, and C is the integral operator arising from the smooth kernel remaining. Relatively standard arguments allow one to handle the compact perturbation once the spline-trig method for B is analyzed. Hence we concentrate on that.

Since we may use the complex exponentials as a basis for T_N, the spline-trig method for the operator B may be written

$$[B(\underset{\sim}{\phi}_\Delta - \underset{\sim}{\phi})]\hat{}(k) = 0, \qquad \frac{1-N}{2} \le k \le \frac{N-1}{2},$$

or

$$\underset{\approx}{m}(k)[\hat{\phi}_\Delta(k)-\hat{\phi}(k)] = 0, \qquad \frac{1-N}{2} \le k \le \frac{N-1}{2}.$$

Since $\underset{\approx}{m}(k)$ is nonsingular, we have

$$\hat{\phi}_\Delta(k) = \hat{\phi}(k), \qquad \frac{1-N}{2} \le k \le \frac{N-1}{2}.$$

Hence the error in $\underset{\sim}{H}^s$ is given by

$$\Big(\sum_{|k|>N/2} |\hat{\phi}_\Delta(k)-\hat{\phi}(k)|^2 \underline{k}^{2s}\Big)^{1/2} \qquad (13)$$

$$\le \Big(\sum_{|k|>N/2} |\hat{\phi}_\Delta(k)|^2 \underline{k}^{2s}\Big)^{1/2} + \Big(\sum_{|k|>N/2} |\phi(k)|^2 \underline{k}^{2s}\Big)^{1/2}.$$

The second term on the right is just the error in the truncated Fourier series approximation to $\underset{\sim}{\phi}$ and is certainly of optimal order. To bound the first term we use the Fourier series characterization of splines:

$$S_d(\Delta) = \{p \in H^0 | \hat{p}(k)k^{d+1} = \hat{p}(k)(-k-N)^{d+1}, k \in \mathbb{Z}\}. \qquad (14)$$

Thus the Fourier coefficients of a spline decay exponentially. Making use of this characterization we can show that the first term on the right of (13) also is optimally small, i.e., $O(h^{d+1-s})$. For details see [1].

6. Test and Trial Functions of Different Parity.

The analysis of the spline-trig method was based on the Fourier series characterization (14) of splines of odd degree on the uniform mesh $\Delta = \{j/N\}_{j \in \mathbb{Z}}$. This characterization remains valid when the degree is even as long the mesh is translated by $h/2$. That is, if we set

$$\Delta = \begin{cases} \{j/N\}_{j\in\mathbb{Z}}, & d \text{ odd}, \\ \{(j+1/2)/N\}_{j\in\mathbb{Z}}, & d \text{ even}, \end{cases}$$

(14) holds, and the analysis of the spline-trig method applies to this case. For example we may use piecewise constant trial functions and trigonometric test functions.

Collocation with piecewise constant or other even degree trial functions is another important numerical method. Here too the meshes for test and trial functions are generally translated one half mesh interval. Otherwise put, the collocation points are the midpoints of the mesh intervals of the trial space.

The analysis of Section 4 does not appear to extend to this case. Instead the only general analysis I know is via Fourier series [9]. A drawback of this approach is that it applies only to operators with convolutional principle part (including, however, the model singular integral equation discussed here) and allows only uniform meshes. However both these restrictions have recently been weakened through additional arguments [5].

One could also analyze via Fourier series a Petrov-Galerkin method employing splines of differing parity with meshes translated by a half interval. Such methods do not appear particularly useful however.

Finally let us mention that G. Schmidt [10] has given an analysis of even degree spline collocation based on and similar in spirit to the analysis in [3] described in Section 4. His analysis applies to collocation by splines of even degree on an arbitrary mesh with collocation points *at the nodes*. As mentioned in Section 3, this method does not work in the trivial case of A = identity, nor does it apply to the strongly elliptic singular integral equation considered here. Schmidt showed, however, that nodal collocation by even degree splines converges with optimal order in appropriate Sobolev spaces for operator equations of the form $AS\phi = f$ where

$$S\phi(s) = \int \cot \pi(s-t)\, \phi(t)dt.$$

In particular this method applies to the first kind singular integral equation

$$(S+K)\phi = f.$$

with K compact.

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