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SUPERCONVERGENCE OF THE GALERKIN APPROXIMATION OF A QUASILINEAR PARABOLIC EQUATION IN A SINGLE SPACE VARIABLE (1)

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Abstract - The asymptotic expansion of the Galerkin solution of a parabolic equation by means of a sequence of elliptic projections that was introduced by Douglas, Dupont, and Wheeder is carried out for a quasilinear equation. This quasi-projection can be applied to establish knot superconvergence in the case of a single space variable. In addition, an optimal order error estimate in $L^\infty(L^\infty)$ is derived for a single space variable.

1. Introduction.

The main result of the paper is the establishment of knot superconvergence of the semidiscrete Galerkin approximation to the solution of a quasilinear parabolic problem in one space variable. The methods used are a direct generalization of those of Douglas-Dupont-Wheeler [3], and the results obtained here are essentially the same as they obtained in the linear case. In particular, the basic tool is the quasi-projection, a sequence of elliptic projections (here with respect to a linearized operator), which approximates the Galerkin solution to high order and furnishes the recipe for initializing the Galerkin procedure.

The quasi-projection is defined in § 4, in arbitrary dimension, and its convergence properties are explored. In § 5, these results are used to establish superconvergence in the single space variable setting. Specifically, we show that when the finite element space consists of continuous, piecewise-polynomial

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functions of degree r subordinate to a quasi-uniform mesh of grid size h, the error at the knots is $0(h^{2r})$, instead of $0(h^{r+1})$, the optimal rate in L^2 or L^∞.

Because of the nonlinearity of the operator, certain quadratic remainder terms are introduced. To bound these, we require optimal order global L^∞ estimates. Such estimates are established, along with the more usual L^2 estimates, in § 6.

2. Preliminaires.

The problem to be considered is given by

$$\text{(2.1)} \quad \begin{cases} c(x,t,u)\dfrac{\partial u}{\partial t} - \nabla\cdot[a(x,t,u)\nabla u + b(x,t,u)] + f(x,t,u) = 0, & \text{on } \Omega\times J, \\ u(x,t)=0, & \text{on } \partial\Omega\times J, \\ u(x,0)=u_0(x), & \text{on } \Omega. \end{cases}$$

Here Ω is a smoothly bounded domain in $\mathbf{R}^d$, $J=[0,T]$, and a, $b=(b_1, b_2, \dots, b_d)$, c and f are bounded, smooth functions on $\Omega\times J\times\mathbf{R}$. We assume that a and c are strictly positive and that f is non-decreasing in u, the last being a condition obtainable without loss of generality by a change of variables.

Much of our error analysis will take place in the usual Sobolev spaces, $W^{s,p}(\Omega)$, s a non-negative integer and $1\le p\le\infty$, with $H^s(\Omega)$ denoting $W^{s,2}(\Omega)$. We shall also use the normed dual of $H^s(\Omega)$, denoted by $H^{-s}(\Omega)$. The norm $\|\cdot\|_{H^s(\Omega)}$ will be abbreviated to $\|\cdot\|_s$, or if $s=0$ simply to $\|\cdot\|$. The notation $\|F\|_{L^p(X)}$ denotes the norm of $t\mapsto\|F(\cdot,t)\|_X$ in $L^p(J)$.

For each of a family of values of h in $(0,1]$ clustering at 0, let $M=M_h\subseteq H_0^1(\Omega)$ be a finite-dimensional space. We assume the following approximation property:

For some positive integer r and any $q\in[1,r+1]$, there is a constant C such that for all $\varphi\in H_0^1(\Omega)\cap H^q(\Omega)$

$$\text{(2.2)} \qquad \inf_{\chi\in M}[\|\varphi-\chi\| + h\|\varphi-\chi\|_1] \le C\|\varphi\|_q h^q.$$

In addition we make the following inverse assumptions:

$$\text{(2.3)} \qquad \|\chi\|_1\le Ch^{-1}\|\chi\| \text{ and } \|\chi\|_{L^\infty}\le Ch^{-\frac{d}{2}}\|\chi\| \text{ for all } \chi\in M.$$

Here and throughout, C denotes a generic constant. As we shall often wish to call attention to the dependence of a constant on u, we adopt the following notation: $C(q, k)$ is a generic constant depending on $\left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^\infty(W^{q,\infty}(\Omega))}$ for $j=0, 1, \dots, k$, but on no higher derivatives of u.

3. The approximate solution.

The solution u of (2.1) satisfies

$$\left(c(u)\frac{\partial u}{\partial t}, v\right)+(a(u)\nabla u+b(u), \nabla v)+(f(u), v)=0, \quad v\in H_0^1(\Omega). \tag{3.1}$$

(The dependence of the coefficients on x and t will frequently be suppressed in the notation.) The approximate solution $U=U_h$: $J\to M$ is then defined by the Galerkin method:

$$\left(c(U)\frac{\partial U}{\partial t}, v\right)+(a(U)\nabla U+b(U), \nabla v)+(f(U), v)=0, \quad v\in M. \tag{3.2}$$

The specification of U at the initial time $t=0$ will be made later.

Let $\zeta=U-u$. For each fixed x and t, we have

$$C(U)\frac{\partial U}{\partial t}=c(u)\frac{\partial u}{\partial t}+c(u)\frac{\partial \zeta}{\partial t}+c_u(u)\frac{\partial u}{\partial t}\zeta+\bar{c}_u\zeta\frac{\partial \zeta}{\partial t}+\bar{c}_{uu}\frac{\partial u}{\partial t}\zeta^2, \tag{3.3a}$$

$$a(U)\nabla U=a(u)\nabla u+a(u)\nabla\zeta+a_u(u)\zeta\nabla u+\bar{a}_u\zeta\nabla\zeta+\bar{a}_{uu}\zeta^2\nabla u, \tag{3.3b}$$

$$b(U)=b(u)+b_u(u)\zeta+\bar{b}_{uu}\zeta^2, \tag{3.3c}$$

$$f(U)=f(u)+f_u(u)\zeta+\bar{f}_{uu}\zeta^2. \tag{3.3d}$$

Here the notations $\bar{F}_u=\int_0^1 F_u(u+\tau\zeta)\,d\tau$ and $\bar{G}_{uu}=\int_0^1 G_{uu}(u+\tau\zeta)(1-\tau)\,d\tau$ apply for $F=a$ or c and $G=a, b, c$ or f. Of course, the subscript notation indicates partial differentiation.

Set $B(\varphi, \psi)=(a(u)\nabla\varphi+[a_u(u)\nabla u+b_u(u)]\varphi, \nabla\psi)+(f_u(u)\varphi, \psi)$.

The Dirichlet problem for the bilinear form B has a unique solution. In fact, if $\varphi \in H_0^1(\Omega)$ and $B(\varphi, \psi)=0$ for all $\psi \in H_0^1(\Omega)$, then

$$\nabla \cdot \{a(u) \nabla \varphi + [a_u(u) \nabla u + b_u(u)] \varphi\} = f_u(u) \varphi;$$

thus, the uniqueness results for equations in divergence form ([1], [6]) imply that $\varphi=0$. It follows ([4]) that elliptic regularity holds in the following form:

$$\text{if } \varphi \in H_0^1(\Omega) \text{ and } B(\mu, \varphi) = (\mu, \psi) \text{ for all } \mu \in H_0^1(\Omega),$$

then

$$\|\varphi\|_{s+2} \leq C \|\psi\|_s,$$

where C depends on $\|a(u)\|_{C^{s+1}(\overline{\Omega})}$, $\|a_u(u) \nabla u + b_u(u)\|_{C^s(\overline{\Omega})}$, and $\|f_u(u)\|_{C^s(\overline{\Omega})}$; i. e., $C=C(s+1, 0)$. Note also that

$$|B(\varphi, \psi)| \leq C(1,0) \|\varphi\|_1 \|\psi\|_1 \text{ for } \varphi, \psi \in H_0^1(\Omega)$$

and

$$B(\varphi, \varphi) \geq \frac{1}{2} (\inf a) \|\varphi\|_1^2 - C(1,0) \|\varphi\|^2 \text{ for } \varphi \in H_0^1(\Omega).$$

The same properties are possessed by the adjoint form $B^*(\varphi, \psi) = B(\psi, \varphi)$. Let

$$R_1 = \overline{c}_u \, \zeta \frac{\partial \zeta}{\partial t} + \overline{c}_{uu} \frac{\partial u}{\partial t} \zeta^2 + \overline{f}_{uu} \, \zeta^2,$$

$$R_2 = \overline{a}_u \, \zeta \nabla \zeta + \overline{a}_{uu} \nabla u \, \zeta^2 + \overline{b}_{uu} \, \zeta^2.$$

Then substituting (3.3) into (3.2) and subtracting (3.1) results in the relation

$$\left(c(u) \frac{\partial \zeta}{\partial t}, v\right) + \left(c_u(u) \frac{\partial u}{\partial t} \zeta, v\right) + B(\zeta, v) = -(R_1, v) - (R_2, \nabla v), \quad v \in M. \tag{3.4}$$

Let $\tilde{u} = \tilde{u}_h \colon J \to M$ be the elliptic projection of u given by

$$B(\tilde{u} - u, v) = 0 \text{ for all } v \in M. \tag{3.5}$$

It is standard that $\tilde{u}$ is uniquely determined for h small (compared to $C(1,0)$). Let $\eta = \tilde{u} - u$, $\xi = \tilde{u} - U$; so, $\zeta = \eta - \xi$. Then from (3.4) we see that

$$\left(c(u)\frac{\partial\xi}{\partial t}+c_u(u)\frac{\partial u}{\partial t}\xi, v\right)+B(\xi, v)= \tag{3.6}$$

$$=\left(c(u)\frac{\partial\eta}{\partial t}+c_u(u)\frac{\partial u}{\partial t}\eta+R_1, v\right)+(R_2, \nabla v),\ v\in M.$$

We shall need to know how B interacts with time differentiation when its stet argument is time-dependent. For this, let $\varphi\colon J\to H_0^1(\Omega)$, $\psi\in H_0^1(\Omega)$. Then by Leibnitz's rule

$$\frac{d^k}{dt^k}B(\varphi,\psi)=\sum_{i=0}^{k}\binom{k}{i}\left\{\left(\left[\frac{d^{k-i}}{dt^{k-i}}a(u)\right]\nabla\frac{\partial^i\varphi}{\partial t^i}, \nabla\psi\right)+\right. \tag{3.7}$$

$$\left.+\left(\frac{d^{k-i}}{dt^{k-i}}[a_u(u)\nabla u+b(u)]\right]\frac{\partial^i\varphi}{\partial t^i}, \nabla\psi\right)+\left(\frac{d^{k-i}}{dt^{k-i}}[f_u(u)]\frac{\partial^i\varphi}{\partial t^i}, \psi\right)\right\}=$$

$$=B\left(\frac{\partial^k\varphi}{\partial t^k}, \psi\right)+\sum_{i=0}^{k-1}D_{ik}\left(\frac{\partial^i\varphi}{\partial t^i}, \psi\right),$$

where for $\gamma\in H_0^1(\Omega)$

$$D_{ik}(\gamma,\psi)=\binom{k}{i}\left\{\left(\left[\frac{d^{k-i}}{dt^{k-i}}a(u)\right]\nabla\gamma+\frac{d^{k-i}}{dt^{k-i}}[a_u(u)\nabla u+b(u)]\,\gamma, \nabla\psi\right)+\right.$$

$$\left.+\left(\frac{d^{k-i}}{dt^{k-i}}[f_u(u)]\,\gamma, \psi\right)\right\}=$$

$$=\binom{k}{i}\left(\gamma, -\nabla\cdot\left(\frac{d^{k-i}}{dt^{k-i}}a(u)\nabla\psi\right)+\frac{d^{k-i}}{dt^{k-i}}[a_u(u)\nabla u+b(u)]\cdot\nabla\psi+\frac{d^{k-i}}{dt^{k-i}}f_u(u)\,\psi\right).$$

Hence,

$$|D_{ik}(\varphi,\psi)|\le\begin{cases}C(1,k-i)\,\|\varphi\|_1\,\|\psi\|_1\,,\\ C(s+1,k-i)\,\|\varphi\|_{-s}\,\|\psi\|_{s+2}\,, & s=0,1,2,\dots.\end{cases} \tag{3.8}$$

The following lemma will prove convenient.

LEMMA 1. Let there be given a linear functional $F\colon H_0^1(\Omega)\to\mathbf{R}$ and numbers $M_1\ge M_2\ge\dots\ge M_{p+1}$, $0\le p\le r$, with

$$|F(\rho)|\le M_s\|\rho\|_s \text{ for all } \rho\in H^s(\Omega)\cap H_0^1(\Omega),\ s=1,2,\dots,p+1.$$

Suppose $\Phi \in H_0^1(\Omega)$ satisfies

$$B(\Phi, \chi) = F(\chi) \text{ for all } \chi \in M.$$

Then there exists $\varepsilon = C(1,0)^{-1} > 0$ such that, for $h < \varepsilon$,

$$\|\Phi\|_{-s} \le C(\max(s,0)+1,0)\,[(M_1 + \inf_{\chi \in M} \|\Phi - \chi\|_1)\,h^{s+1} + M_{s+2}],$$

$$\text{for } s = -1, 0, \dots, p-1.$$

PROOF. First note that

$$B(\Phi,\Phi) = \inf_{\chi \in M} [B(\Phi, \Phi - \chi) + F(\chi - \Phi) + F(\Phi)] \le$$

$$\le (C(1,0)\,\|\Phi\|_1 + M_1) \inf_{\chi \in M} \|\chi - \Phi\|_1 + M_1 \|\Phi\|_1 .$$

Hence

$$\|\Phi\|_1 \le C(1,0)\,(\|\Phi\| + \inf_{\chi \in M} \|\chi - \Phi\|_1) + CM_1 . \tag{3.9}$$

Next, given $\psi \in H^s(\Omega)$, $0 \le s \le p-1$, define $\varphi \in H^{s+2}(\Omega) \cap H_0^1(\Omega)$ by

$$B(\rho, \varphi) = (\rho, \psi) \text{ for all } \rho \in H_0^1(\Omega);$$

thus,

$$\|\varphi\|_{s+2} \le C(s+1,0)\,\|\psi\|_s .$$

Then,

$$(\Phi, \psi) = B(\Phi, \varphi) = \inf_{\chi \in M} [B(\Phi, \varphi - \chi) + F(\chi - \varphi) + F(\varphi)] \le \tag{3.10}$$

$$\le C(s+1,0)\,[(\|\Phi\|_1 + M_1)\,h^{s+1} + M_{s+2}]\,\|\psi\|_s .$$

Taking $s=0$ and $\psi = \Phi$ yields the inequality

$$\|\Phi\| \le C(1,0)\,[(\|\Phi\|_1 + M_1)\,h + M_2].$$

Substituting this result in (3.9) gives, for small h,

$$\|\Phi\|_1 \le C(1,0)\,(M_1 + \inf_{\chi \in M} \|\Phi - \chi\|_1),$$

which is the desired estimate for $s = -1$.

Using this estimate in conjunction with (3.10) completes the proof.

THEOREM 2. Let $k \geq 0$ and $1 \leq q \leq r+1$. There exists $\varepsilon > 0$ depending on $\|u\|_{L^\infty(W^{1,\infty}(\Omega))}$ such that, if $h < \varepsilon$, then

$$\left\| \frac{\partial^k \eta}{\partial t^k} \right\|_{-s} \leq C(\max(q, s+1), k)\, h^{s+q} \text{ for } -1 \leq s \leq r-1.$$

PROOF. For $\chi \in M$,

$$B\left(\frac{\partial^k \eta}{\partial t^k}, \chi\right) = F_k(\chi) \equiv \sum_{i=0}^{k-i} D_{ik}\left(\frac{\partial^i \eta}{\partial t^i}, \chi\right).$$

When $k=0$, $F_k=0$. Also,

$$\inf_{\chi \in M} \|\eta - \chi\|_1 = \inf_{\chi \in M} \|u-\chi\|_1 \leq C(\mathrm{q}, 0)\, h^{q-1}.$$

Lemma 1 can be invoked to establish the theorem for $k=0$.

We proceed by induction on k. For k positive and for all $\rho \in H^s(\Omega) \cap H_0^1(\Omega)$,

$$|F_k(\rho)| \leq C(\max(s,0)+1, k) \sum_{i=0}^{k-1} \left\| \frac{\partial^i \eta}{\partial t^i} \right\|_{-s} \|\rho\|_{s+2} \leq$$

$$\leq C(\max(q, s+1), k)\, h^{s+q} \|\rho\|_{s+2}, \quad s = -1, 0, \dots, r-1,$$

where (3.8) was used to obtain the first inequality and the inductive hypothesis for the second. Also,

$$\inf_{\chi \in M} \left\| \frac{\partial^k \eta}{\partial t^k} - \chi \right\|_1 = \inf_{\chi \in M} \left\| \frac{\partial^k u}{\partial t^k} - \chi \right\|_1 = C(q, k)\, h^{q-1}.$$

Combining these results with Lemma 1 completes the proof.

4. The quasi-projection.

Set $z_0 = \eta$ and let $z_j = z_{j,h}: J \to M$, $j = 1, 2, \dots$, be defined recursively by

$$B(z_j, v) = -\left(c(u)\frac{\partial z_{j-1}}{\partial t} + c_u(u)\frac{\partial u}{\partial t} z_{j-1}, v\right), \quad v \in M. \tag{4.1}$$

The essential property of the z_j's is that they decrease geometrically in size.

THEOREM 3. Let $j \geq 0$, $k \geq 0$, and $1 \leq q \leq r+1$. There exists $\varepsilon = C(1, 0)^{-1} > 0$

such that

$$(4.2)\qquad \left\| \frac{\partial^k z_j}{\partial t^k} \right\|_{-s} \leq C\,(\max(q, \max(s,0)+2j+1), k+j)\, h^{s+q+2j}$$

for $-1 \leq s \leq r-1-2j$ and $h < \varepsilon$.

PROOF. The case $j=0$ is covered by Theorem 2. We proceed by induction on j.

For $j>0$ and $k \geq 0$, let

$$F(\rho) = -\sum_{i=0}^{k-1} D_{ik}\left(\frac{\partial^i z_j}{\partial t^i}, \rho\right) - \left(\frac{d^k}{dt^k}\left[c(u)\frac{\partial z_{j-1}}{\partial t} + c_u(u)\frac{\partial u}{\partial t} z_{j-1}\right], \rho\right)$$

for $\rho \in H_0^1(\Omega)$. Then

$$B\left(\frac{\partial^k z_j}{\partial t^k}, \chi\right) = F(\chi) \text{ for all } \chi \in M,$$

and

$$|F(\rho)| \leq \left\{ C\,(\max(s,0)+1, k) \sum_{i=0}^{k-1} \left\| \frac{\partial^i z_j}{\partial t^i} \right\|_{-s} + \right.$$

$$\left. + C\,(s+2, k+1) \sum_{i=0}^{k+1} \left\| \frac{\partial^i z_{j-1}}{\partial t^i} \right\|_{-s-2} \right\} \|\rho\|_{s+2}$$

for $s=-1, 0, \ldots$. The inductive hypothesis implies that F fulfills the hypotheses of Lemma 1 with $p=r-2j$, and

$$M_{s+2} \leq C\,(\max(q, \max(s,0)+2j+1), k+j)\left(h^{s+q+2j} + \sum_{i=0}^{k-1} \left\| \frac{\partial^i z_j}{\partial t^i} \right\|_{-s}\right)$$

for $s=-1, 0, \ldots, r-2j-1$. Since the infimum appearing in the conclusion of the lemma is zero for $\Phi = \dfrac{\partial^k z_j}{\partial t^k}$, it follows that

$$\left\| \frac{\partial^k z_j}{\partial t^j} \right\|_{-s} \leq C\,(\max(q, \max(s,0)+2j+1), k+j)\left(h^{s+q+2j} + \sum_{i=0}^{k-1} \left\| \frac{\partial^i z_j}{\partial t^i} \right\|_1 h^{s+1} + \right.$$

$$\left. + \sum_{i=0}^{k-1} \left\| \frac{\partial^i z_j}{\partial t^i} \right\|_{-s} \right).$$

If $k=0$, the sums are trivial, and the claimed result holds. The theorem can then be completed by a simple induction on k.

To complete the description of our Galerkin method, we impose the following initial condition on U:

$$U(0)=\tilde{u}(0)+z_1(0)+\ldots+z_k(0) \tag{4.3}$$

for some $k\leq\left[\frac{r-1}{2}\right]$. We shall see that the most useful value of k is $\left[\frac{r-1}{2}\right]$. Let $\theta_k=\xi+z_1+\ldots+z_k$. Then for all $v\in M$

$$\begin{cases} \left(C(u)\frac{\partial\theta_k}{\partial t}+c_u(u)\frac{\partial u}{\partial t}\theta_k, v\right)+B(\theta_k, v)= \\ \qquad =\left(c(u)\frac{\partial z_k}{\partial t}+c_u(u)\frac{\partial u}{\partial t}z_k+R_1, v\right)+(R_2, \nabla v), \\ \theta_k(0)=0. \end{cases} \tag{4.4}$$

This is a direct consequence of (3.6) and the definition of the z_j's.

THEOREM 4. If $2k\leq r-1$ and $1\leq q\leq r+1$ and if $h<C(1,0)^{-1}$ is sufficiently small, then

$$||\theta_k||_{L^\infty(L^2)}+||\theta_k||_{L^2(H^1)}\leq \tag{4.5}$$

$$\leq C(\max(q, 2k+2), k+1)\, h^{q+\min(2k+1,r-1)}+C(1,1)\,(||R_1||_{L^2(H^{-1})}+||R_2||_{L^2(L^2)}).$$

PROOF. Choosing $v=\theta_k$ in (4.4) leads to the relation

$$\frac{1}{2}\frac{d}{dt}(c(u)\theta_k,\theta_k)+B(\theta_k,\theta_k)=\frac{1}{2}\left(\left[\frac{d}{dt}c(u)\right]\theta_k,\theta_k\right)-\left(c_u(u)\frac{\partial u}{\partial t}\theta_k,\theta_k\right)+$$

$$+\left(c(u)\frac{\partial z_k}{\partial t}+c_u(u)\frac{\partial u}{\partial t}z_k,\theta_k\right)+(R_1,\theta_k)+(R_2,\nabla\theta_k).$$

Thus,

$$\frac{d}{dt}(c(u)\theta_k,\theta_k)+||\theta_k||_1^2\leq C(1,1)\,||\theta_k||^2+$$

$$+C(1,1)\left(||z_k||_{-1}^2+\left\|\frac{\partial z_k}{\partial t}\right\|_{-1}^2\right)+C\,(||R_1||_{-1}^2+||R_2||^2).$$

The Gronwall inequality implies that

$$\|\theta_k\|_{L^\infty(L^2)}+\|\theta_k\|_{L^2(H^1)}\leq$$

$$\leq C(1,1)\left(\|z_k\|_{L^2(H^{-1})}+\left\|\frac{\partial z_k}{\partial t}\right\|_{L^2(H^{-1})}+\|R_1\|_{L^2(H^{-1})}+\|R_2\|_{L^2(L^2)}\right),$$

and (4.5) follows from (4.2).

THEOREM 5. If $1\leq q\leq r+1$ and if $h<C(1,0)^{-1}$ is sufficiently small, then

$$\|\theta_k\|_{L^\infty(H^1)}+\left\|\frac{\partial\theta_k}{\partial t}\right\|_{L^2(L^2)}\leq \tag{4.6}$$

$$\leq\begin{cases} C(\max(q,2k+2),k+1)h^{q+2k}+C(1,1)\left(\|R_1\|_{L^2(L^2)}+h^{-1}\|R_2\|_{L^2(L^2)}\right), & \text{if } 2k\leq r-1,\\ C(\max(q,2k+2),k+2)\,h^{q+2k+1}+C(1,1)\left(\|R_1\|_{L^2(L^2)}+h^{-1}\|R_2\|_{L^2(L^2)}\right), & \text{if } 2k\leq r-2.\end{cases}$$

PROOF. Note that

$$B\left(\theta_k,\frac{\partial\theta_k}{\partial t}\right)=\frac{1}{2}\frac{d}{dt}(a(u)\nabla\theta_k,\nabla\theta_k)-\frac{1}{2}\left(\frac{d}{dt}[a(u)]\nabla\theta_k\right)+$$

$$+\left([a_u(u)\nabla u+b_u(u)]\,\theta_k,\nabla\frac{\partial\theta_k}{\partial t}\right)+\left(f_u(u)\,\theta_k,\frac{\partial\theta_k}{\partial t}\right).$$

Consequently, if the test function v in (4.4) is taken to be $\partial\theta_k/\partial t$, it follows that

$$\left(c(u)\frac{\partial\theta_k}{\partial t},\frac{\partial\theta_k}{\partial t}\right)+\frac{1}{2}\frac{d}{dt}(a(u)\nabla\theta_k,\nabla\theta_k)= \tag{4.7}$$

$$=-\left(c_u(u)\frac{\partial u}{\partial t}\,\theta_k,\frac{\partial\theta_k}{\partial t}\right)+\frac{1}{2}\left(\frac{d}{dt}[a(u)]\nabla\theta_k,\nabla\theta_k\right)-$$

$$-\left([a_u(u)\nabla u+b_u(u)]\,\theta_k,\nabla\frac{\partial\theta_k}{\partial t}\right)-\left(f_u(u)\,\theta_k,\frac{\partial\theta_k}{\partial t}\right)+$$

$$+\left(c(u)\frac{\partial z_k}{\partial t}+c_u(u)\frac{\partial u}{\partial t}z_k,\frac{\partial\theta_k}{\partial t}\right)+\left(R_1,\frac{\partial\theta_k}{\partial t}\right)+\left(R_2,\nabla\frac{\partial\theta_k}{\partial t}\right)=$$

$$=J_1+J_2+\ldots+J_7\,.$$

Let $\varepsilon = \inf c\,(x, t, z)/7$. The following estimates are clear:

(4.8)
$$|J_1| \leq \varepsilon \left\|\frac{\partial \theta_k}{\partial t}\right\| + C\,(0, 1)\,\|\theta_k\|^2,$$
$$|J_2| \leq C\,(0, 1)\,\|\theta_k\|_1^2,$$
$$|J_4| \leq \varepsilon \left\|\frac{\partial \theta_k}{\partial t}\right\| + C\,\|\theta_k\|^2,$$
$$|J_5| \leq \varepsilon \left\|\frac{\partial \theta_k}{\partial t}\right\| + C\,(0, 1)\left[\left\|\frac{\partial z_k}{\partial t}\right\| + \|z_k\|^2\right],$$
$$|J_6| \leq \varepsilon \left\|\frac{\partial \theta_k}{\partial t}\right\|^2 + C\,\|R_1\|^2.$$

As a result of the inverse assumption (2.3),

$$|J_7| \leq \|R_2\| \left\| \nabla \frac{\partial \theta_k}{\partial t} \right\| \leq \varepsilon \left\|\frac{\partial \theta_k}{\partial t}\right\|^2 + Ch^{-2}\,\|R_2\|^2. \tag{4.9}$$

To bound the integral of J_3 with respect to time, first integrate by parts in time. Thus,

$$\begin{aligned}
\left|\int_0^t J_3\, d\tau\right| &= \Big| -([a_u(u)\,\nabla u + b_u(u)]\,\theta_k, \nabla\theta_k)\,(t) + \\
&+ \int_0^t \left(\frac{d}{dt}\,[a_u(u)\,\nabla u + b_u(u)]\,\theta_k, \nabla\theta_k\right) d\tau + \\
&+ \int_0^t \left([a_u(u)\,\nabla u + b_u(u)]\,\frac{\partial \theta_k}{\partial t}, \nabla\theta_k\right) d\tau \Big| \leq \\
&\leq \frac{\inf a}{2}\,\|\theta_k(t)\|_1^2 + C\,(1, 1)\,\|\theta_k\|^2_{L^\infty(L^2)} + \\
&+ C\,(1, 0)\,\|\theta_k\|^2_{L^2(H^1)} + \varepsilon \int_0^t \left\|\frac{\partial \theta_k}{\partial t}\right\|^2 d\tau.
\end{aligned} \tag{4.10}$$

Combining (4.7)-(4.10), we obtain the bound

$$(4.11)\qquad \int_0^t \left\|\frac{\partial \theta_k}{\partial t}\right\|^2 d\tau + \|\theta_k(t)\|_1^2 \leq C(1,1)\left(\|\theta_k\|^2_{L^2(H^1)} + \|\theta_k\|^2_{L^\infty(L^2)}\right) +$$

$$+C(0,1)\left(\|z_k\|^2_{L^2(L^2)} + \left\|\frac{\partial z_k}{\partial t}\right\|^2_{L^2(L^2)}\right) +$$

$$+C\cdot\left(\|R_1\|^2_{L^2(L^2)} + h^{-2}\|R_2\|^2_{L^2(L^2)}\right).$$

The upper inequality of (4.6) follows from Theorems 3 and 4 and (4.11), where $2k \leq r-1$.

If $2k \leq r-2$, we obtain the improved estimate by integrating the time integral of J_5 by parts before applying Theorems 3 and 4:

$$\left|\int_0^t j_5\, d\tau\right| = \left|\left(c(u)\frac{\partial z_k}{\partial t} + c_u(u)\frac{\partial u}{\partial t} z_k, \theta_k\right)(t) -\right.$$

$$-\int_0^t \left(\left[\frac{d}{dt}\, c(u)\right]\frac{\partial z_k}{\partial t} + \frac{d}{dt}\left[c_u(u)\frac{\partial u}{\partial t}\right] z_k, \theta_k\right) d\tau -$$

$$\left. -\int_0^t c(u)\, \frac{\partial^2 z_k}{\partial t^2} + c_u(u)\frac{\partial u}{\partial t}\frac{\partial z_k}{\partial t}, \theta_k\right) d\tau \right| \leq$$

$$\leq \frac{\inf a}{4}\|\theta_k(t)\|_1^2 + C(1,2)\|\theta_k\|^2_{L^2(H^1)} +$$

$$+C(1,1)\left(\|z_k\|^2_{L^\infty(H^{-1})} + \left\|\frac{\partial z_k}{\partial t}\right\|^2_{L^\infty(H^{-1})} + \left\|\frac{\partial^2 z_k}{\partial t^2}\right\|^2_{L^\infty(H^{-1})}\right).$$

5. The quasi-projection in a single space variable.

We now specialize to the case $\Omega=(0,1)$ with the finite element spaces being piecewise polynomial functions. Thus, for each h we suppose given a partition $0=x_0^h<x_1^h<\ldots<x_{N_h}^h=1$ such that $\max\limits_{j=1,\ldots,N_h}(x_j^h-x_{j-1}^h)=h$ and

$$\max_{\substack{j=1,\dots,Nh\\ i=1,\dots,Nh}} ((x_j^h - x^h_{j-1})(x_i^h - x^h_{i-1})^{-1} \leq C,$$

with C independent of h. (This last condition is referred to as quasi-uniformity). It is supposed that M_h is the space of continuous functions on I which restrict to polynomials of degree at most r on each (x^h_{j-1}, x_j^h). We remark that the dimension of M_h can be decreased without affecting the arguments below by requiring that the functions of M_h have derivatives up to order p_j for some $p_j < r$ at certain knots x_j^h; however, at any knot at which superconvergence is to take place the smoothness constraint on M_h must reduce to continuity. The hypotheses (2.2) and (2.3) hold for these choices for M_h.

The inequality (where $\zeta = U - u$, as before)

$$\text{(5.1)} \qquad \|\zeta\|_{L^\infty(L^\infty)} + \left\|\frac{\partial \zeta}{\partial t}\right\|_{L^2(L^2)} + h\,\|\zeta\|_{L^2(H^1)} \leq$$

$$\leq C\,(\max(q, 2k+1), \max(k, 1))\, h^q$$

for $1 \leq q \leq r+1$ will be proved in § 6 in the case of a single space variable. Since the initial condition $U(0)$ depends on the choice of k, ζ depends on k. An immediate corollary of (5.1) is the inequality

$$\text{(5.2)} \qquad \|R_1\|_{L^2(L^2)} + h\,\|R_2\|_{L^2(L^2)} \leq$$

$$\leq C\,(\max(q, 2k+1), \max(k, 1))\, h^{2q}, \quad 1 \leq q \leq r+1.$$

We now use the method of [3] to extract superconvergence results from Theorem 5. Let $\bar{x} \in (0, 1)$ be a knot in each of the partitions; i. e., for each h there exists $i(h)$ so that $\bar{x} = x^h_{i(h)}$. Let

$$\tilde{H}^s = \{u\colon u\,|_{(0,\bar{x})} \in H^s((0, \bar{x})),\ u|_{(\bar{x},1)} \in H^s((\bar{x}, 1))\} \times \mathbf{R},$$

and norm $\tilde{H}^s$ by

$$|||(u, \beta)|||^2 = \|u\|^2_{H^s((0,\bar{x}))} + \|u\|^2_{H^s((\bar{x},1))} + \beta^2.$$

For the pair (u, β) and (v, γ) of elements of $\tilde{H}^0$, define the inner product

$$[(u, \beta), (v, \gamma)] = (u, v) + \beta\gamma.$$

Finally, for $z \in H^1(I)$ and $s \geq 0$, let

$$|||z|||_{-s} = \sup_{|||(u,\beta)|||_s=1} [(z, z(\bar{x})), (u, \beta)].$$

Note that

$$|z(\bar{x})| \leq |||z|||_{-s}, \quad s=0, 1, 2, \dots . \tag{5.3}$$

THEOREM 6. Let $1 \leq q \leq r+1$ and $0 \leq s \leq r-2j-1$. Then for $h < C(1,0)^{-1}$ sufficiently small and $t \in J$,

$$\left|\left|\left| \frac{\partial^k z_j}{\partial t^k} \right|\right|\right|_{-s} \leq C(\max(q, s+2j+1), j+k)\, h^{q+s+2j}. \tag{5.4}$$

PROOF. If $(\psi, \beta) \in \widetilde{H}^s$, determine $\varphi \in H_0^1(I)$ for each $t \in J$ by

$$-(a(u)\varphi')' + [a_u(u)\, u' + b_u(u)]\, \varphi' + f_u(u)\, \varphi = \psi \quad \text{on } I \backslash \{\bar{x}\},$$

$$a(u)\, \varphi' \Big|_{\bar{x}-0}^{\bar{x}+0} = -\beta.$$

$\left(\text{Here we write } \varphi' \text{ for } \dfrac{\partial x}{\partial \varphi}\right)$. Then, $B(\mu, \varphi) = [(\mu, \mu(\bar{x})), (\psi, \beta)]$ for all $\mu \in H_0^1(I)$, and

$$|||(\varphi, \varphi(\bar{x}))|||_{s+2} \leq C(s+1, 0)\, |||(\psi, \beta)|||_s$$

(with constant independent of $\bar{x}$). Also note that, if $\rho \in W^{s+1,\infty}(I)$ and $\mu \in H_0^1(I)$,

$$(\rho\mu', \varphi') = \rho(\bar{x})\, \mu(\bar{x})\, \varphi'(\bar{x}-0) - \rho(\bar{x})\, \mu(\bar{x})\, \varphi'(\bar{x}+0) - $$
$$- \int_0^{\bar{x}} \mu\, (\rho\varphi')'\, dx - \int_{\bar{x}}^1 \mu\, (\rho\varphi')'\, dx.$$

Thus,

$$|(\rho\mu', \varphi')| \leq \frac{1}{a(\bar{x}, u(\bar{x}))}\, |\rho(\bar{x})\, \mu(\bar{x})\, \beta| + |(\mu, (\rho\varphi')')| \leq \tag{5.5}$$
$$\leq C\, |||\mu|||_{-s}\, |||(\psi, \beta)|||_s,$$

where the constant depends on $||\rho||_{W^{s+1,\infty}}$ (but on no higher derivatives).

To prove the theorem, we proceed by an outer induction on j and an inner induction on k. For $j=0$ and $v \in M$,

$$\left[\left(\frac{\partial^k \eta}{\partial t^k}, \frac{\partial^k \eta(\bar{x})}{\partial t^k}\right), (\psi, \beta)\right] = B\left(\frac{\partial^k \eta}{\partial t^k}, \varphi - v\right) + \sum_{i=0}^{k-1} D_{ik}\left(\frac{\partial^i \eta}{\partial t^i}, \varphi - v\right) - \sum_{i=0}^{k-1} D_{ik}\left(\frac{\partial^i \eta}{\partial t^i}, \varphi\right). \tag{5.6}$$

The assumption that the elements of M are continuous, but not necessarily differentiable, at $\bar{x}$ implies that

$$\inf_{v \in M} ||\varphi - v||_1 \leq C\, |||(\varphi, \varphi(\bar{x}))|||_{s+2}\, h^{s+1} \text{ for } 0 \leq s \leq r-1;$$

hence, by (5.5) and (5.6),

$$\left|\left|\left|\frac{\partial^k \eta}{\partial t^k}\right|\right|\right|_{-s} \leq C(s+1, 0)\left|\left|\frac{\partial^k \eta}{\partial t^k}\right|\right|_1 h^{s+1} + \sum_{i=0}^{k-1} C(s+1, k-i)\left|\left|\frac{\partial^i \eta}{\partial t^i}\right|\right|_1 h^{s+1} + \sum_{i=0}^{k-1} C(s+1, k-i)\left|\left|\left|\frac{\partial^i \eta}{\partial t^i}\right|\right|\right|_{-s}. \tag{5.7}$$

The case $j=0$ follows from Theorem 2 and the inductive hypothesis.

For $j>0$ the proof is similar. For $v \in M$,

$$\left[\left(\frac{\partial^k z_j}{\partial t^k}, \frac{\partial^k z_j(\bar{x})}{\partial t^k}\right), (\psi, \beta)\right] = B\left(\frac{\partial^k z_j}{\partial t^k}, \varphi - v\right) + \left(\frac{d^k}{dt^k}\left[c(u)\frac{\partial z_{j-1}}{\partial t} + c_u(u)\frac{\partial u}{\partial t} z_{j-1}\right], \varphi - v\right) + \sum_{i=0}^{k-1} D_{ik}\left(\frac{\partial^i z_j}{\partial t^i}, \varphi - v\right) - \left(\frac{d^k}{dt^k}\left[c(u)\frac{\partial z_{j-1}}{\partial t} + c_u(u)\frac{\partial u}{\partial t} z_{j-1}\right], \varphi\right) - \sum_{i=0}^{k-1} D_{ik}\left(\frac{\partial^i z_j}{\partial t^i}, \varphi\right).$$

Consequently,

$$\left|\left|\left|\frac{\partial^k z_j}{\partial t^k}\right|\right|\right|_{-s} \leq C(s+1, 0)\left|\left|\frac{\partial^k z_j}{\partial t^k}\right|\right|_1 h^{s+1} + C(s+1, k+1)\left(\left|\left|\frac{\partial z_{j-1}}{\partial t}\right|\right|_{-1} + ||z_{j-1}||_{-1}\right) h^{s+1} + \sum_{i=0}^{k-1} C(s+1, k-i)\left|\left|\frac{\partial^i z_j}{\partial t^i}\right|\right|_1 h^{s+1} + \tag{5.8}$$

$$+C\,(s+1,k+1)\sum_{i=0}^{k+1}\left|\!\left|\!\left|\frac{\partial^i z_{j-1}}{\partial t^i}\right|\!\right|\!\right|_{-s-2}+\sum_{i=0}^{k-1}C\,(s+1,k-i)\left|\!\left|\!\left|\frac{\partial^i z_j}{\partial t^i}\right|\!\right|\!\right|_{-s}.$$

Then Theorem 6 follows from (5.8) and Theorem 3 by induction.

Inequalities (5.3) and (5.4) imply that

$$|z_j(\bar{x},t)|\le C\,(\max(q,s+2j+1),j)\,h^{q+s+2j},\quad j=0,1,\dots, \tag{5.9}$$

for $0\le s\le r-2j-1$, $1\le q\le r+1$, and h small. Now, write $u-U$ in the form

$$(u-U)(\bar{x},t)=\theta_k(\bar{x},t)-\sum_{j=0}^{k} z_j(\bar{x},t). \tag{5.10}$$

Since $|\theta_k(\bar{x},t)|\le\|\theta_k(\cdot,t)\|_1$, (4.13) and (5.2) show that

$$|\theta_k(\bar{x},t)|\le\begin{cases}C\,(\max(q,2k+2),k+1)\,(h^{q+2k}+h^{2q-2}), & \text{if } 2k\le r-1,\\ C\,(\max(q,2k+2),k+2)\,(h^{q+2k+1}+h^{2q-2}), & \text{if } 2k\le r-2,\end{cases} \tag{5.11}$$

again for $1\le q\le r+1$. Thus, (5.9), (5.10) and (5.11) imply the following theorem.

THEOREM 7. Let $1\le q\le r+1$ and $0\le k\le\left[\frac{1}{2}(r-1)\right]$. Then for $h<C\,(1,0)^{-1}$ sufficiently small,

$$|(u-U)(\bar{x},t)|\le\begin{cases}C\,(\max(q,2k+2),k+1)\,(h^{q+2k}+h^{2q-2}), & \text{if } 2k\le r-2,\\ C\,(\max(q,2k+2),k+2)\,(h^{q+2k+1}+h^{2q-2}), & \text{if } 2k\le r-1,\end{cases} \tag{5.12}$$

where $U(x,0)=\tilde{u}(x,0)+z_1(x,0)+\dots+z_k(x,0)$ and $\bar{x}=x_{i(h)}$ is a knot at which the smoothness of M_h reduces to continuity. If $k=\left[\frac{1}{2}(r-1)\right]$ and $q=r+1$, then

$$|(u-U)(\bar{x},t)|\le\begin{cases}C\left(r+1,\frac{1}{2}(r+1)\right)h^{2r}, & r \text{ odd},\\ C\left(r+1,\frac{1}{2}r+1\right)h^{2r}, & r \text{ even}.\end{cases} \tag{5.13}$$

Theorem 7, the principal objective of this paper, is a direct generalization of Theorem 6.1 of [3]; the constants in (5.12) and (5.13) depend on the same derivatives of the solution u as did the constants for the linear parabolic equation with time-independent coefficients treated in [3]. Naturally, the dependence of the constants on these derivatives is less precisely defined in the nonlinear case. Notice that the dependence is balanced with respect to differentiation with respect to space and time for odd r and for even r is as near so as can be obtained without consideration of fractional order differentiation in time.

For the linear problem treated in [3] Thomée [5] has recently obtained the $0(h^{2r})$-convergence rate at (x, t) for $t \geq \delta > 0$ with the initial values $U(x, 0)$ being the L^2-projection of u_0.

6. Global estimates.

The main goal of this section is the proof of (5.1). Along the way we shall demonstrate most of the commonly encountered L^2 and L^∞ estimates for parabolic problems. For the L^∞ estimates we rely on the estimates for the two-point boundary value problem found in [2]; consequently, we shall work mostly in one space dimension. However, the L^2 estimates work equally well in higher dimensions, and we begin by working under the assumptions set out in the first paragraph of § 2.

Following Wheeler ([7], [8]), we separate $U-u$ into two parts, one representing the difference between U and an appropriate elliptic projection of u and the other representing the error in the elliptic projection. Define the projection $\hat{u}: J \to M$ by

$$(a(u) \nabla (\hat{u}-u), \nabla v) = 0 \text{ for all } v \in M. \tag{6.1}$$

Note that the projection (6.1) differs from that of (3.5). Let

$$\mu = \hat{u} - u, \quad \psi = \hat{u} - U, \tag{6.2}$$

where U remains defined by (3.2) and (4.3). It is easily seen by subtracting (3.5) from (6.1), setting the test function equal to $\hat{u}-\tilde{u}$, and using (3.10) that

$$\|\tilde{u}-\hat{u}\|_1 \leq C(1, 0) \|\tilde{u}-u\| \leq C(q, 0) h^q \tag{6.3}$$

for $t \in J$ and $1 \leq q \leq r+1$. Since $\psi(0) = (\hat{u}-\tilde{u})(0) - z_1(0) - \dots - z_k(0)$, it follows from (4.2) and (6.3) that

$$\|\psi(0)\|_1 \leq C(\max(q, 2k+1), k) h^q, \quad 1 \leq q \leq r+1. \tag{6.4}$$

It is standard that

$$\|\mu\|_{-s} \leq C\,(s+1, 0)\, \|u\|_q\, h^{q+s}, \ t\in J,\ 1\leq q\leq r+1,\ -1\leq s\leq r-1. \tag{6.5}$$

Since differentiation of (6.1) with respect to t leads to the relation

$$\left(a\,(u)\, \nabla \frac{\partial \mu}{\partial t}, \nabla v\right) = -\left(\frac{d}{dt}\, a\,(u)\, \Big] \nabla \mu, \nabla v\right),\ v\in M,\ t\in J, \tag{6.6}$$

it follows from (6.5) that

$$\left\| \frac{\partial u}{\partial t} \right\|_1 \leq C\,(0, 1) \left\{ \|u\|_q + \left\| \frac{\partial \mu}{\partial t} \right\|_q \right\} h^{q-1} \tag{6.7}$$

for $t\in J$ and $1\leq q\leq r+1$. If $\beta\in H^s(\Omega)$ with $0\leq s\leq r-1$ and $\alpha\in H_0^1(\Omega)\cap H^{s+2}(\Omega)$ is determined by $(a\,(u)\,\nabla v, \nabla\alpha)=(v, \beta)$ for $v\in H_0^1(\Omega)$, it can be deduced from the relation

$$\left(\frac{\partial \mu}{\partial t}, \beta\right) = \left(a\nabla \frac{\partial \mu}{\partial t}, \nabla\,(\alpha-\chi)\right) + \left(\left[\frac{d}{dt}\, a\,(u)\right] \nabla \mu,\ \ \nabla\,(\alpha-\chi)\right)$$

$$+\left(\mu, \nabla\cdot\left(\left[\frac{d}{dt} \mathrm{a}\,(u)\right] \nabla\alpha\right)\right),$$

which holds for all $\chi\in M$, that, for $t\in J$,

$$\left\| \frac{\partial \mu}{\partial t} \right\|_{-r} \leq C\,(s+1, 1) \left[\left\| \frac{\partial u}{\partial t} \right\|_q + \|u\|_q \right] h^{q+s},\ \ 1\leq q\leq r+1. \tag{6.8}$$

Next, we shall derive some estimates for ψ. Since

$$\left(C\,(u)\, \frac{\partial \hat{u}}{\partial t}, v\right) + (a\,(u)\, \nabla \hat{u} + b\,(u), \nabla v) + (f\,(u), v) = \left(c\,(u) \frac{\partial \mu}{\partial t}, v\right)$$

for $v\in M$, subtracting (3.2) from this relation produces

$$\left(c\,(U) \frac{\partial \psi}{\partial t}, v\right) + (a\,(U)\, \nabla\psi, \nabla v) = \tag{6.9}$$

$$= \left(c\,(u) \frac{\partial \mu}{\partial t}, v\right) + ([a\,(U) - a\,(u)]\ \nabla \hat{u} + b\,(U) - b\,(u), \nabla v) +$$

$$+\left([c\,(U) - c\,(u)]\, \frac{\partial \hat{u}}{\partial t}\ + f\,(U) - f\,(u), v\right),\ v\in M.$$

Recall [7] that

$$(6.10)\qquad \left(c(U)\frac{\partial\psi}{\partial t},\psi\right)=$$

$$=\frac{d}{dt}\int_{\Omega}\int_{0}^{\psi}c(\hat{u}-\tau)\,\tau\,d\tau\,dx-\int_{\Omega}\int_{0}^{\psi}\left[\frac{d}{dt}c(\hat{u}-\tau)\right]\tau\,d\tau\,dx.$$

Thus, if v is taken to be the function ψ in (6.9),

$$(6.11)\qquad \frac{d}{dt}\int_{\Omega}\int_{0}^{\psi}c(\hat{u}-\tau)\,\tau\,d\tau\,dx+(a(U)\nabla\psi,\nabla\psi)\le$$

$$\le\frac{1}{2}(\inf a)\,\|\nabla\psi\|^2+C\left\|\frac{\partial\mu}{\partial t}\right\|^2+C\left(\|\hat{u}\|_{W^{1,\infty}},\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty}\right)\|u-U\|^2+$$

$$+C\left(\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty}\right)\|\psi\|^2.$$

Integrating (6.11) in time leads to the inequality

$$(6.12)\qquad \|\psi\|_{L^\infty(L^2)}+\|\psi\|_{L^2(H^1)}\le$$

$$\le C\left(\|\hat{u}\|_{L^\infty(W^{1,\infty})},\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty(L^\infty)}\right)\left(\left\|\frac{\partial\mu}{\partial t}\right\|_{L^2(L^2)}+\|\mu\|_{L^2(L^2)}+\|\psi(0)\|\right)\le$$

$$\le C(\max(q,2k+1),k)\,C\left(\|\hat{u}\|_{L^\infty(W^{1,\infty})},\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty(L^\infty)}\right)h^q$$

for $1\le q\le r+1$, where (6.4), (6.5), (6.8), the inequality $\|u-U\|\le\|\psi\|+\|\mu\|$, and the equivalence of $\|\psi\|^2$ and the double integral of the first term of (6.11) have been used; the equivalence follows from the boundedness of $c(x,t,z)$ above and below by positive numbers.

Now choose $v=\partial\psi/\partial t$ in (6.9). A calculation leads to the inequality

$$(6.13)\qquad \left(c(U)\frac{\partial\psi}{\partial t},\frac{\partial\psi}{\partial t}\right)+\frac{1}{2}\frac{d}{dt}(a(U)\nabla\psi,\nabla\psi)\le$$

$$\le\frac{1}{4}(\inf c)\left\|\frac{\partial\psi}{\partial t}\right\|^2+C\left(\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty}\right)\left(\|\psi\|_1^2+\|\mu\|^2+\left\|\frac{\partial\mu}{\partial t}\right\|^2\right)+$$

$$+\left([a(U)-a(u)]\,\nabla\hat{u}+b(U)-b(u),\nabla\frac{\partial\psi}{\partial t}\right)+$$

$$+\frac{1}{2}\left(a_u(U)\frac{\partial U}{\partial t}\nabla\psi,\nabla\psi\right).$$

By the inverse hypothesis (2.3),

$$(6.14)\qquad \left|\frac{1}{2}\left(a_u(U)\frac{\partial U}{\partial t}\nabla\psi,\nabla\psi\right)\right|\leq C\cdot\left(\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty}+\left\|\frac{\partial\psi}{\partial t}\right\|_{L^\infty}\right)\|\psi\|_1^2\leq$$

$$\leq\frac{1}{4}(\inf c)\left\|\frac{\partial\psi}{\partial t}\right\|^2+C\cdot\left(\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty}\|\psi\|_1^2+h^{-d}\|\psi\|_1^4\right).$$

Next, by integration by parts in time,

$$(6.15)\qquad \int_0^t[a(U)-a(u)]\,\nabla\hat{u},\ \nabla\frac{\partial\psi}{\partial t}\Big)d\tau=I_1+I_2+I_3+I_4,$$

where

$$(6.16)\qquad I_1=([a(U)-a(u)]\,\nabla u,\nabla\psi)|_0^t\leq$$

$$\leq\frac{1}{3}(\inf a)\|\nabla\psi(t)\|^2+\|\psi(0)\|_1^2+C(\|\hat{u}\|_{L^\infty(W^{1,\infty})})(\|\mu\|^2_{L^\infty(L^2)}+\|\psi\|^2_{L^\infty(L^2)}),$$

$$(6.17)\qquad I_2=-\int_0^t([a_t(U)-a_t(u)]\,\nabla\hat{u},\nabla\psi)\,d\tau\leq$$

$$\leq\|\psi\|^2_{L^2(H^1)}+C(\|\hat{u}\|_{L^\infty(W^{1,\infty})})(\|\mu\|^2_{L^2(L^2)}+\|\psi\|^2_{L^2(L^2)}),$$

$$(6.18)\qquad I_3=-\int_0^t\left[\left(a_u(U)\frac{\partial U}{\partial t}-a_u(u)\frac{\partial u}{\partial t}\right]\nabla\hat{u},\nabla\psi\right)d\tau=$$

$$=-\int_0^t\left(\left\{a_u(U)\frac{\partial(U-u)}{\partial t}+[a_u(U)-a_u(u)]\frac{\partial u}{\partial t}\right\}\nabla\hat{u},\nabla\psi\right)d\tau\leq$$

$$\leq \frac{1}{8}(\inf c)\left\|\frac{\partial\psi}{\partial t}\right\|^2_{L^2(L^2)}+\left\|\frac{\partial\mu}{\partial t}\right\|^2_{L^2(L^2)}+C(0,1)\,\|\mu\|^2_{L^2(L^2)}+$$

$$+C(0,1)\,\|\hat{u}\|^2_{L^\infty(W^{1,\infty})}\|\psi\|^2_{L^2(H^1)},$$

and

$$(6.19)\qquad I_4=-\int_0^t\left([a(U)-a(u)]\,\nabla\frac{\partial\hat{u}}{\partial t},\nabla\psi\right)d\tau\leq$$

$$\leq\|\psi\|^2_{L^2(H^1)}+C\left(\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty(W^{1,\infty})}\right)(\|\mu\|^2_{L^2(L^2)}+\|\psi\|^2_{L^2(L^2)}).$$

The term in (6.13) involving the coefficient b can be handled similarly. Thus,

$$(6.20)\qquad \int_0^t\left\|\frac{\partial\psi}{\partial t}\right\|^2 d\tau+\|\psi(t)\|_1^2\leq$$

$$\leq C(1,1)\,C\left(\|\hat{u}\|_{L^\infty(W^{1,\infty})},\left\|\frac{\partial\hat{u}}{\partial t}\right\|_{L^\infty(W^{1,\infty})}\right)(\|\psi(0)\|_1^2+\|\psi\|^2_{L^\infty(L^2)}+\|\psi\|^2_{L^2(H^1)}$$

$$+\|\mu\|^2_{L^\infty(L^2)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(L^2)}\Big)+Ch^{-d}\int_0^t\|\psi\|_1^4\,d\tau.$$

By Gronwall,

$$(6.21)\qquad \left\|\frac{\partial\psi}{\partial t}\right\|^2_{L^2(L^2)}+\|\psi\|^2_{L^\infty(H^1)}\leq$$

$$\leq C(1,1)\,C(\|\hat{u}\|_{W^{1,\infty}(\Omega\times J)})\exp\{Ch^{-d}\|\psi\|^2_{L^2(H^1)}\}\cdot$$

$$\cdot\left[\|\psi(0)\|_1^2+\|\psi\|^2_{L^\infty(L^2)}+\|\psi\|^2_{L^2(H^1)}+\|\mu\|^2_{L^\infty(L^2)}+\left\|\frac{\partial\mu}{\partial t}\right\|^2_{L^\infty(L^2)}\right].$$

Let $q\geq\frac{1}{2}d$. Then, (6.12) implies that the exponential term is bounded. Apply (6.4), (6.5), (6.8), and (6.12) to (6.21); thus,

(6.22)
$$\left\|\frac{\partial\psi}{\partial t}\right\|_{L^2(L^2)}+\|\psi\|_{L^\infty(H^1)}\leq$$
$$\leq C\,(\max\,(q,2k+1),k)\;C\,(\|\hat{u}\|_{W^{1,\infty}(\Omega\times J)})\,h^q$$

for $\frac{1}{2}d\leq q\leq r+1$.

In order to obtain L^∞-estimates we return to the case $\Omega=I=(0,1)$, and we also readopt the additional assumptions on M_4 made at the outset of § 5. The following result combines Lemmas 3.2 and 4.2 of [2].

LEMMA 8. Let $z\in H_0^1(I)$ satisfy

$$(a\,(u)\,z',v')=0,\quad v\in M.$$

Then,

$$\|z\|_{W^{1,\infty}}\leq C\,(1,0)\inf_{\chi\in M}\|z-\chi\|_{W^{1,\infty}}.$$

An immediate consequence of Lemma 8 and (6.1) is that

(6.23)
$$\|\mu\|_{W^{1,\infty}}\leq C\,(1,0)\inf_{\chi\in M}\|u-\chi\|_{W^{1,\infty}}\leq C\,(q,0)\,h^{q-1}$$

for $1\leq q\leq r+1$. In particular, this implies that

(6.24)
$$\|\hat{u}\|_{L^\infty(W^{1,\infty})}\leq C\,(1,0).$$

Next, let ρ: $J\to H_0^1(I)$ be determined by

$$(a\,(u)\,\rho',v')=-\left(\frac{da\,(u)}{\partial t}\mu',v'\right),\quad v\in H_0^1(I).$$

An easy calculation shows that

$$\|\rho\|_{W^{1,\infty}}\leq C\,(0,1)\,\|\mu\|_{W^{1,\infty}},\quad t\in J.$$

Then (6.6) and another application of Lemma 8 with $z=\frac{\partial\mu}{\partial t}-\rho$ lead to the inequality

(6.25)
$$\left\|\frac{\partial\mu}{\partial t}\right\|_{W^{1,\infty}}\leq C\,(q,1)\,h^{q-1},\quad t\in J,\ 1\leq q\leq r+1.$$

For $q=1$,

$$\left\| \frac{\partial \hat{u}}{\partial t} \right\|_{L^\infty(W^{1,\infty})} \leq C(1,1). \tag{6.26}$$

The following theorem states the complete set of L^∞-estimates for μ and $\partial\mu/\partial t$; the $L^\infty(W^{1,\infty})$-estimates appear above in (6.23) and (6.25).

THEOREM 9. Let $1 \leq q \leq r+1$. Then,

$$\|\mu\|_{L^\infty(L^\infty)} + h\|\mu\|_{L^\infty(W^{1,\infty})} \leq C(q,0)\, h^q \tag{6.27}$$

and

$$\left\| \frac{\partial\mu}{\partial t} \right\|_{L^\infty(L^\infty)} + h \left\| \frac{\partial\mu}{\partial t} \right\|_{L^\infty(W^{1,\infty})} \leq C(q,1)\, h^q. \tag{6.28}$$

PROOF. A duality argument that is valid in the single space variable case will be employed. Let $g \in L^1(I)$ and let $G\colon J \to W^{2,1}(I)$ satisfy

$$-(a(u)\,G')' = g, \quad x \in I,$$

$$G=0, \quad x \in \partial I.$$

Since $\dim(I)=1$,

$$\|G\|_{W^{2,1}} \leq C(1,0)\,\|g\|_{L^1}, \quad t \in J.$$

Then, for $\chi \in M$,

$$(\mu, g) = (a(u)\,\mu', G' - \chi'),$$

and

$$\|\mu\|_{L^\infty} \leq C(1,0)\,\|\mu\|_{W^{1,\infty}}\, h, \quad t \in J.$$

Thus, (6.27) has been proved.

Similarly,

$$\left(\frac{\partial\mu}{\partial t}, g\right) = \left(a(u)\left(\frac{\partial\mu}{\partial t}\right)', G'-\chi'\right) + \left(\frac{d}{dt}[a(u)]\,\mu', G'-\chi'\right) +$$

$$+ \left(\mu', \left(\frac{d}{dt}[a(u)]\,G'\right)'\right).$$

Thus,

$$\left\| \frac{\partial \mu}{\partial t} \right\|_{L^\infty} \leq C(1,1) \left\{ h \left\| \frac{\partial \mu}{\partial t} \right\|_{W^{1,\infty}} + h \|\mu\|_{W^{1,\infty}} + \|\mu\|_{L^\infty} \right\} \leq C(q,1)\, h^q,$$

as was to be shown.

Finally, the inequality (5.1) can be proved.

THEOREM 10. Let $\zeta = U - u$, where u is the solution of (2.1) and U is the solution of (3.2) and (4.3) for some k such that $0 \leq 2k \leq r-1$. Then, for $1 \leq q \leq r+1$,

$$\|\zeta\|_{L^\infty(L^\infty)} + \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^2(L^2)} + h \|\zeta\|_{L^\infty(H^1)} \leq \tag{5.1}$$

$$\leq C(\max(q, 2k+1), \max(k,1))\, h^q.$$

PROOF. Note that $\zeta = \mu - \psi$. It follows from (6.22) and (6.26) that

$$\left\| \frac{\partial \psi}{\partial t} \right\|_{L^2(L^2)} + \|\psi\|_{L^\infty(H^1)} \leq C(\max(q, 2k+1), \max(k,1))\, h^q. \tag{6.29}$$

Hence, (5.1) follows from Theorem 9 and (6.29).

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