Name: ANSWERS

CSE/MATH 455 Exam I October 7, 1996 Professor Arnold

15 pts 1. Suppose that x^* is a fixed point of $g(x)$, that $g'(x^*) = 0$, and that $|g''(x)|$ is continuous and bounded. Starting from some x_0, define a sequence x_n by $x_{n+1} = g(x_n)$. Prove that

$$|e_{n+1}| \leq M|e_n|^2,$$

where $e_n = x_n - x^*$. Give an expression for M.

By Taylor's theorem

$$g(x) = g(x^*) + g'(x^*)(x - x^*) + \frac{1}{2}g''(\xi)(x - x^*)^2 = x^* + \frac{1}{2}g''(\xi)(x - x^*)^2$$

for some ξ. Thus

$$e_{n+1} = x_{n+1} - x^* = g(x_n) - x^* = \frac{1}{2}g''(\xi)(x_n - x^*)^2 = \frac{1}{2}g''(\xi)e_n^2.$$

Taking absolute values gives

$$|e_{n+1}| \leq M|e_n|^2$$

where $M = \frac{1}{2}\max_{\xi}|f''(\xi)|$.

15 pts 2. Let x_1, x_2, and x_3 be three distinct points and y_1, y_2, and y_3 three values. Let $p(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating the values y_1 and y_2 at the points x_1 and x_2, and let $q(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating y_2 and y_3 at x_2 and x_3. Prove that

$$r(x) = \frac{x_3 - x}{x_3 - x_1} p(x) + \frac{x - x_1}{x_3 - x_1} q(x)$$

is the Lagrange interpolating polynomial of degree ≤ 2 interpolating y_1, y_2, and y_3 at x_1, x_2, and x_3.

We need to check that $r(x)$ is a polynomial of degree ≤ 2 and the $r(x_i) = y_i$ for $i = 1, 2, 3$. That the degree of $r(x) \leq 2$ is obvious, since the degrees of $p(x)$ and $q(x)$ are ≤ 1. Further, since $p(x_1) = y_1$, $p(x_2) = q(x_2) = y_2$, and $q(x_3) = y_3$,

$$\begin{aligned}
r(x_1) &= \frac{x_3 - x_1}{x_3 - x_1} p(x_1) + \frac{x_1 - x_1}{x_3 - x_1} q(x_1) = 1 \cdot y_1 + 0 \cdot q(x_1) = y_1 \qquad \checkmark \\
r(x_2) &= \frac{x_3 - x_2}{x_3 - x_1} p(x_2) + \frac{x_2 - x_1}{x_3 - x_1} q(x_2) = \frac{x_3 - x_2}{x_3 - x_1} y_2 + \frac{x_2 - x_1}{x_3 - x_1} y_2 = y_2 \qquad \checkmark \\
r(x_3) &= \frac{x_3 - x_3}{x_3 - x_1} p(x_3) + \frac{x_3 - x_1}{x_3 - x_1} q(x_3) = 0 \cdot p(x_3) + 1 \cdot y_3 = y_3 \qquad \checkmark
\end{aligned}$$

15 pts 3. Write down the quadratic polynomial which interpolates the values of $\sqrt{x}$ at the points $x = 1, 4, 9$, using Newton's form of the interpolating polynomial.

We start by constructing the divided difference table for the points $1, 4, 9$, *and the values* $1 = \sqrt{1}$, $2 = \sqrt{4}$, $3 = \sqrt{9}$*:*

$$
\begin{array}{cccc}
1 & 1 & & \\
 & & \dfrac{2-1}{4-1} = \dfrac{1}{3} & \\
4 & 2 & & \dfrac{1/5 - 1/3}{9-1} = -\dfrac{1}{60} \\
 & & \dfrac{3-2}{9-4} = \dfrac{1}{5} & \\
9 & 3 & &
\end{array}
$$

Thus Newton's form of the interpolating polynomial is

$$p(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{60}(x-1)(x-4).$$

15 pts 4. Write down the formula for Newton's method applied to the equation $x^3 - \cos x = 0$.

The formula for Newton's method is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, *or, in this case*

$$x_{n+1} = x_n - \frac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n}.$$

11 pts 5. Which of the following sequences appears to be converging to 1 quadratically?

a) 0.8027893, 0.9810498, 0.9980254, 0.9998018

b) 0.8027893, 0.9601194, 0.9984095, 0.9999975 ✓

c) 0.8027893, 0.9013942, 0.9500143, 0.9749988

d) 0.8027893, 0.9993521, 0.9992412, 0.9999999

11 pts 6. Recall that IEEE single precision uses a 24 bit mantissa and an exponent running from -126 to 127. What is the smallest floating point number in this system which is greater than 10?

a) $10 + 10^{-7}$

b) $10 + 2^{-20}$ ✓

c) $10 + 2^{-23}$

d) $10 + 2^{-24}$

18 pts 7. For each of the following statements, fill in the blank with one of the following: "the bisection method", "Newton's method", "the secant method", or "Muller's method".

a) <u>Newton's method</u> converges quickly if you start close to a root of the function, but it requires that you be able to compute the derivative of the function.

b) <u>The bisection method</u> requires that you know a bracketing interval.

c) <u>The secant method</u> has an order of convergence of about 1.6.

d) <u>Muller's method</u> is based on quadratic interpolation.

e) The errors in <u>the secant method</u> obey $|e_{n+1}| \approx M|e_n||e_{n-1}|$.

f) <u>Muller's method</u> may return a complex number as an approximation to the root, even if all the previous approximations are real numbers.