Introduction to Divided Differences

Note: assignments will be graded on clarity of the presentation as well as mathematical correctness. Be neat, clear, and precise!

Let $f(x)$ be a function defined on an interval I. If x_0 and x_1 are two distinct points in I, define the divided difference $f[x_0, x_1]$ by the formula

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The following theorem is an immediate consequence of the mean value theorem.

Theorem 1. *If $f(x)$ is continuously differentiable on an interval I containing x_0 and x_1, then there exists a point $\xi \in I$ such that*

$$f[x_0, x_1] = f'(\xi).$$

The next theorem shows that if x_1 is close to x_0, then $f[x_0, x_1]$ is close to $f'(x_0)$, at least if $f''(x)$ is not too big on I.

Theorem 2. *If $f(x)$ is twice continuously differentiable on an interval I containing x_0 and $x_0 + h$ for some number h, then*

$$|f[x_0, x_0 + h] - f'(x_0)| \le M|h|,$$

where $M = \frac{1}{2} \max_{x \in I} |f''(x)|$.

Proof. By Taylor's theorem

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2} f''(\xi)h^2,$$

for some ξ between x_0 and $x_0 + h$. Rearranging this equation gives

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) = \frac{1}{2} f''(\xi)h,$$

or,

$$f[x_0, x_0 + h] - f'(x_0) = \frac{1}{2} f''(\xi)h.$$

Thus

$$|f[x_0, x_0 + h] - f'(x_0)| = \frac{1}{2}|f''(\xi)||h| \le M|h|. \quad \square$$

Typeset by $\mathcal{AMS}$-TEX

1. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| \le Nh^2.$$

Give a specific expression for N.

2. If three distinct points x_0, x_1, and x_2 all belong to I, then the *second divided difference*, $f[x_0, x_1, x_2]$, is defined as

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}.$$

Show that $f[x_0, x_1, x_2]$ is symmetric in its three arguments; in other words, that reordering the arguments doesn't change the value: $f[x_0, x_1, x_2] = f[x_0, x_2, x_1] = f[x_2, x_1, x_0] = \dots$.

3. Show that

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(\zeta)$$

for some ζ between the smallest and largest of x_0, x_1, and x_2. State precisely the necessary hypotheses. (Hint: you may want to prove and use this lemma: if $g(x)$ is continuous on an interval I and a and b are positive numbers, then for any ξ, $\zeta \in I$, there is a point $\eta \in I$ such that $ag(\xi) + bg(\zeta) = (a + b)g(\eta)$.)

4. Find a simple expression for $f[x_0 - h, x_0, x_0 + h]$.

5. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0, x_0 + h/2] - \frac{1}{2} f''(x_0)| \le Ph^2.$$

Give a specific expression for P.