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# Fixed points and fixed point iteration

Let $g(x)$ be a continuous function defined for $x \in \mathbb{R}$. A *fixed point* of $g$ is a solution of the equation $x = g(x)$. Thus a fixed point of $g$ is the $x$-value of a point where the graph $y = g(x)$ crosses the line $y = x$. (Don't mix up a fixed point of $g$ with a root of $g$, which is a solution to the equation $g(x) = 0$, or a point where the graph of $g$ crosses the $x$-axis.) If we are given an algebraic equation, we may always write it in the form of a fixed point equation; in fact we can do so in many different ways.

1. Write the equation $x^3 + 3x = \cos(x)$ as a fixed point equation $x = g(x)$ in *three* different ways.

2. Now suppose that we choose $x_0$ arbitrarily and define $x_1$, $x_2$, ... by the one point iteration:

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots .$$

(This iteration is also called "successive substitution" or "fixed point iteration.") Show that *if* $x^* = \lim_{n\to\infty} x_n$ exists, then $x^*$ is a fixed point of $g(x)$. Thus, if the one point iteration converges to something, then that something is a fixed point.

3. Suppose that the function $g(x)$ maps an interval $[a, b]$ into itself (that is, $g(x) \in [a, b]$ for all $x \in [a, b]$). Prove that the function $g(x)$ must have a fixed point in $[a, b]$. (Hint: Use the Intermediate Value Theorem.)

4. Again suppose the function $g(x)$ maps the interval $[a, b]$ into itself, and suppose also that $g$ is a *contraction* on $[a, b]$. This means that there is a number $\lambda < 1$ such that

$$|g(x) - g(y)| \le \lambda |x - y| \text{ for all } x, y \in [a, b].$$

Prove that in this case $g(x)$ has *exactly one* fixed point in $[a, b]$.

5. Under the same assumptions as the previous problem show that the one point iteration above converges to the fixed point for any choice of $x_0 \in [a, b]$. What is the order and rate of convergence?

6. We will now explore fixed point iteration for quadratic functions of the form $g(x) = cx(1-x)$ where $c$ is a constant. (Of course we don't need a computer to solve this equation, but by looking at different values of $c$ we can see almost all the behaviors that come up in realistic fixed point problems.) If $1 < c \leq 4$, then this function maps the unit interval $[0, 1]$ into itself. Show that there are exactly two fixed points in the unit interval in this case. What are they exactly?

7. Use the graphical Matlab program `fixedpoint` to explore the behavior of fixed point iteration for this quadratic for various values of $c$ in $(1, 4]$ and various initial values $x_0$. Determine as best you can for which values of $c$ and $x_0$ the iteration converges to the positive fixed point. For which values does it converge to the fixed point at 0? For which values does it not converge to either? For which value of $c$ is the convergence fastest? For which values is the convergence monotonic (i.e., increasing up to $x^*$ or decreasing down to $x^*$).

8. Suppose that $g(x^*) = x^*$ and that $|g'(x^*)| < 1$ (assume that $g'$ exists and is continuous). Then $x^*$ is called an "attractive fixed point." Prove that in this case there is a number $\epsilon > 0$ so that $g$ is a contraction on the interval $[x^* - \epsilon, x^* + \epsilon]$. Conclude that if $x^*$ is an attractive fixed point and $x_0$ is chosen sufficiently close to $x^*$, then the one point iteration converges to $x^*$.

9. If $x^*$ is a fixed point but $|g'(x^*)| > 1$, then $x^*$ is called a "repulsive fixed point." Show that for a repulsive fixed point there is a number $\epsilon > 0$ such that if $x_n$ lies in the interval $[x^* - \epsilon, x^* + \epsilon]$ (but is not exactly the fixed point $x^*$), then $x_{n+1}$ is actually further from $x^*$ than $x_n$ is. In other words, one point iteration does *not* converge to repulsive fixed points.

10. Newton's method to find a root of $f(x)$ takes the form of an iteration $x_{n+1} = g(x_n)$. Show that the root of $f$ is an attractive fixed point of $g$.

11. If $x_0$ is sufficiently near a fixed point $x^*$ of $g(x)$, it may happen that the further iterates (a) converge monotonically to $x^*$, (b) converge to $x^*$ alternating between larger and smaller values than $x^*$, or (c) don't converge. Based on your work above, how can you predict which of these will occur in terms of the value of $g'(x^*)$?