Introduction to Divided Differences

1. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| \le Nh^2.$$

Give a specific expression for N.

Answer:

Theorem. *Suppose that $f(x)$ is three times continuously differentiable on an interval I containing $x_0 - h/2$ and $x_0 + h/2$ for some numbers x_0 and h. Then*

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| \le Nh^2.$$

where $N = \frac{1}{24}\max_{x\in I}|f'''(x)|$.

Proof. By Taylor's theorem

$$f(x_0 + h/2) = f(x_0) + f'(x_0)\frac{h}{2} + \frac{f''(x_0)}{2}\left(\frac{h}{2}\right)^2 + \frac{f'''(\xi)}{6}\left(\frac{h}{2}\right)^3$$

for some number ξ between x_0 and $x_0 + h/2$. Similarly

$$\begin{aligned} f(x_0 - h/2) &= f(x_0) + f'(x_0)\frac{-h}{2} + \frac{f''(x_0)}{2}\left(\frac{-h}{2}\right)^2 + \frac{f'''(\xi')}{6}\left(\frac{-h}{2}\right)^3 \\ &= f(x_0) - f'(x_0)\frac{h}{2} + \frac{f''(x_0)}{2}\left(\frac{h}{2}\right)^2 - \frac{f'''(\xi')}{6}\left(\frac{h}{2}\right)^3, \end{aligned}$$

for some number ξ' between x_0 and $x_0 - h/2$.

Subtracting these two equations gives

$$f(x_0 + h/2) - f(x_0 - h/2) = f'(x_0)h + \frac{h^3}{48}[f'''(\xi) + f'''(\xi')],$$

and dividing by h then gives

$$\frac{f(x_0 + h/2) - f(x_0 - h/2)}{h} = f'(x_0) + \frac{h^2}{48}[f'''(\xi) + f'''(\xi')].$$

The left hand side is exactly the divided difference $f[x_0 - h/2, x_0 + h/2]$, so we have shown that

$$f[x_0 - h/2, x_0 + h/2] - f'(x_0) = \frac{h^2}{48}[f'''(\xi) + f'''(\xi')].$$

Taking absolute values and using the triangle inequality we get

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| = \left|\frac{h^2}{48}\right| |f'''(\xi) + f'''(\xi')| \le \left|\frac{h^2}{48}\right| [|f'''(\xi)| + |f'''(\xi')|].$$

Clearly

$$|f'''(\xi)| \le \max_{x\in I}|f'''(x)| \qquad \text{and} \qquad |f'''(\xi')| \le \max_{x\in I}|f'''(x)|,$$

so

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| \le \left|\frac{h^2}{48}\right| 2\max_{x\in I}|f'''(x)| = \frac{1}{24}\max_{x\in I}|f'''(x)|h^2 = Nh^2. \quad \square$$

2. If three distinct points x_0, x_1, and x_2 all belong to I, then the *second divided difference*, $f[x_0, x_1, x_2]$, is defined as

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}.$$

Show that $f[x_0, x_1, x_2]$ is symmetric in its three arguments; in other words, that reordering the arguments doesn't change the value: $f[x_0, x_1, x_2] = f[x_0, x_2, x_1] = f[x_2, x_1, x_0] = \dots$.

Answer:
(1)

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{\frac{f(x_1)-f(x_0)}{x_1-x_0} - \frac{f(x_2)-f(x_1)}{x_2-x_1}}{x_0 - x_2} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_0 - x_2)} - \frac{f(x_2) - f(x_1)}{(x_2 - x_1)(x_0 - x_2)} \\ &= -\frac{f(x_0)}{(x_1 - x_0)(x_0 - x_2)} + f(x_1)\left[\frac{1}{(x_1 - x_0)(x_0 - x_2)} + \frac{1}{(x_2 - x_1)(x_0 - x_2)}\right] \\ &\qquad - \frac{f(x_2)}{(x_2 - x_1)(x_0 - x_2)}. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{(x_1 - x_0)(x_0 - x_2)} + \frac{1}{(x_2 - x_1)(x_0 - x_2)} &= \frac{(x_2 - x_1) + (x_1 - x_0)}{(x_1 - x_0)(x_0 - x_2)(x_2 - x_1)} \\ &= \frac{x_2 - x_0}{(x_1 - x_0)(x_0 - x_2)(x_2 - x_1)} = \frac{1}{(x_1 - x_0)(x_1 - x_2)}. \end{aligned}$$

Thus (1) becomes

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_0)}.$$

Thus the divided difference is the sum of three terms, one for each of the points x_0, x_1, and x_2: for each point we have f evaluated at that point, divided by the product of the difference between that point and each of the other two. If we reorder the points, this just reorders the three terms, but the sum stays the same. □

3. Show that

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(\zeta)$$

for some ζ between the smallest and largest of x_0, x_1, and x_2. State precisely the necessary hypotheses. (Hint: you may want to prove and use this lemma: if $g(x)$ is continuous on an interval I and a and b are positive numbers, then for any ξ, $\zeta \in I$, there is a point $\eta \in I$ such that $ag(\xi) + bg(\zeta) = (a + b)g(\eta)$.)

Answer: First the lemma.

Lemma. *If $g(x)$ is continuous on an interval I and a and b are positive numbers, then for any ξ, $\zeta \in I$, there is a point $\eta \in I$ such that $ag(\xi) + bg(\zeta) = (a + b)g(\eta)$.*

Proof. Obviously

$$\min_{x\in I} g(x) \le g(\xi) \le \max_{x\in I} g(x).$$

Since a is positive

$$a \min_{x\in I} g(x) \le ag(\xi) \le a \max_{x\in I} g(x).$$

Similarly

$$b \min_{x\in I} g(x) \le bg(\eta) \le b \max_{x\in I} g(x).$$

Adding these two inequalities and dividing by $a + b$ gives

$$\min_{x\in I} g(x) \le \frac{ag(\xi) + bg(\zeta)}{a + b} \le \max_{x\in I} g(x).$$

By the Intermediate Value Theorem, there exists η between a and b for which

$$g(\eta) = \frac{ag(\xi) + bg(\zeta)}{a + b}.$$

This completes the proof of the lemma. □

To obtain the result of the problem, we use Taylor's theorem to expand $f(x_0)$ about x_1:

$$f(x_0) = f(x_1) + f'(x_1)(x_0 - x_1) + \frac{f''(\zeta)}{2}(x_0 - x_1)^2$$

for some ζ between x_0 and x_1. Subtracting $f(x_1)$ from both sides and dividing by $x_0 - x_1$ gives

$$f[x_0, x_1] = f'(x_1) + \frac{f''(\zeta)}{2}(x_0 - x_1). \tag{2}$$

Similarly

$$f[x_2, x_1] = f'(x_1) + \frac{f''(\xi)}{2}(x_2 - x_1) \tag{3}$$

for some ξ between x_1 and x_2. Subtracting equation (2) from equation (3) and dividing by $x_2 - x_0$ gives

$$f[x_0, x_1, x_2] = \frac{1}{2}\frac{f''(\xi)(x_2 - x_1) + f''(\zeta)(x_1 - x_0)}{x_2 - x_0}.$$

We can write the right hand side as

$$ag(\xi) + bg(\zeta)$$

where

$$a = \frac{1}{2}\frac{x_2 - x_1}{x_2 - x_0}, \quad b = \frac{1}{2}\frac{x_1 - x_0}{x_2 - x_0}.$$

and $g = f''$. Since the order of the points doesn't affect the divided difference, we can assume that $x_0 < x_1 < x_2$ (otherwise switch the points, which won't change $f[x_0, x_1, x_2]$). Under this assumption, a and b are positive and $a + b = 1/2$, so the lemma gives $f[x_0, x_1, x_2] = 1/2 f''(\eta)$ for some η as desired.

4. Find a simple expression for $f[x_0 - h, x_0, x_0 + h]$.

Answer:

$$f[x_0 - h, x_0, x_0 + h] = \frac{\frac{f(x_0)-f(x_0-h)}{h} - \frac{f(x_0+h)-f(x_0)}{h}}{-2h} = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{2h^2}.$$

5. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0, x_0 + h/2] - \frac{f''(x_0)}{2}| \leq Ph^2.$$

Give a specific expression for P.

Answer: Again use Taylor's theorem, this time keeping four terms:

$$f(x_0 + h/2) = f(x_0) + f'(x_0)\frac{h}{2} + \frac{f''(x_0)}{2}\left(\frac{h}{2}\right)^2 + \frac{f'''(x_0)}{6}\left(\frac{h}{2}\right)^3 + \frac{f^{(iv)}(\xi)}{24}\left(\frac{h}{2}\right)^4,$$

$$f(x_0 - h/2) = f(x_0) - f'(x_0)\frac{h}{2} + \frac{f''(x_0)}{2}\left(\frac{h}{2}\right)^2 - \frac{f'''(x_0)}{6}\left(\frac{h}{2}\right)^3 + \frac{f^{(iv)}(\xi')}{24}\left(\frac{h}{2}\right)^4,$$

Therefore

$$f(x_0 - h/2) - 2f(x_0) + f(x_0 + h/2) = f''(x_0)\left(\frac{h}{2}\right)^2 + \frac{1}{24}\left(\frac{h}{2}\right)^4 [f^{(iv)}(\xi) + f^{(iv)}(\xi')],$$

and so (using the formula from the last exercise)

$$\begin{aligned} f[x_0 - h/2, x_0, x_0 + h/2] &= \frac{f(x_0 - h/2) - 2f(x_0) + f(x_0 + h/2)}{2(h/2)^2} \\ &= \frac{1}{2}f''(x_0) + \frac{1}{192}h^2[f^{(iv)}(\xi) + f^{(iv)}(\xi')]. \end{aligned}$$

Thus

$$|f[x_0 - h/2, x_0, x_0 + h/2] - \frac{f''(x_0)}{2}| \leq Ph^2$$

with $P = \frac{1}{96}\max_{x \in I} |f^{(iv)}(x)|$. (Here I is the interval from $x_0 - h/2$ to $x_0 + h/2$.)