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CSE/MATH 455, Arnold — Lab 2 — September 19, 1996

Fixed points and fixed point iteration

Let $g(x)$ be a continuous function defined for $x \in \mathbb{R}$. A *fixed point* of g is a solution of the equation $x = g(x)$. Thus a fixed point of g is the x-value of a point where the graph $y = g(x)$ crosses the line $y = x$. (Don't mix up a fixed point of g with a root of g, which is a solution to the equation $g(x) = 0$, or a point where the graph of g crosses the x-axis.) If we are given an algebraic equation, we may always write it in the form of a fixed point equation; in fact we can do so in many different ways.

1. Write the equation $x^3 + 3x = \cos(x)$ as a fixed point equation $x = g(x)$ in *three* different ways.

Answer: There are many possibilities, e.g.,

$$x = \frac{\cos(x) - x^3}{3},$$
$$x = \cos(x) - x^3 - 2x,$$
$$x = \sqrt[3]{\cos(x) - 3x}.$$

2. Now suppose that we choose x_0 arbitrarily and define $x_1, x_2, \ldots$ by the one point iteration:

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots .$$

(This iteration is also called "successive substitution" or "fixed point iteration.") Show that *if* $x^* = \lim_{n\to\infty} x_n$ exists, then x^* is a fixed point of $g(x)$. Thus, if the one point iteration converges to something, then that something is a fixed point.

Answer: We will take limits on both sides of the equation $x_{n+1} = g(x_n)$. Since we are supposing that $\lim_{n\to\infty} x_n = x^*$, we also have $\lim_{n\to\infty} x_{n+1} = x^*$, and, since $g(x)$ is continuous, that $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n) = g(x^*)$. Therefore we get $x^* = g(x^*)$, i.e., x^* is a fixed point of $g(x)$.

3. Suppose that the function $g(x)$ maps an interval $[a, b]$ into itself (that is, $g(x) \in [a, b]$ for all $x \in [a, b]$). Prove that the function $g(x)$ must have a fixed point in $[a, b]$. (Hint: Use the Intermediate Value Theorem.)

Answer: Since $a \leq g(x) \leq b$ for all x, $a \leq g(a)$ and $g(b) \leq b$. Therefore the function $x - g(x)$ is ≤ 0 when $x = a$ and ≥ 0 when $x = b$. By the intermediate value theorem, $x - g(x) = 0$ for some x in $[a, b]$. I.e., x is a fixed point.

4. Again suppose the function $g(x)$ maps the interval $[a, b]$ into itself, and suppose also that g is a *contraction* on $[a, b]$. This means that there is a number $\lambda < 1$ such that

$$|g(x) - g(y)| \leq \lambda |x - y| \text{ for all } x,\ y \in [a, b].$$

Prove that in this case $g(x)$ has *exactly one* fixed point in $[a, b]$.

Answer: We know there is at least one fixed point by the previous problem, so we show that there cannot be two different fixed points. If x and y are both fixed points, then

$$|x - y| = |g(x) - g(y)| \leq \lambda |x - y|.$$

Since $\lambda < 1$ this can happen only if $|x - y| = 0$, i.e., $x = y$. This shows that there cannot be two different fixed points.

5. Under the same assumptions as the previous problem show that the one point iteration above converges to the fixed point for any choice of $x_0 \in [a, b]$. What is the order and rate of convergence?

Answer: Since $x_{n+1} = g(x_n)$ and $x^* = g(x^*)$, we have

$$|x_{n+1} - x^*| = |g(x_n) - g(x^*)| \leq \lambda |x_n - x^*|.$$

Applying this inequality repeatedly we have

$$|x_n - x^*| \leq \lambda |x_{n-1} - x^*| \leq \lambda^2 |x_{n-2} - x^*| \leq \ldots \lambda^n |x_0 - x^*|.$$

This implies that $\lim_{n \to \infty} |x_n - x^*| = 0$, i.e., that x_n converges to x^*. It also shows that the convergence is linear with rate λ. By problem 2, x^* is a fixed point.

6. We will now explore fixed point iteration for quadratic functions of the form $g(x) = cx(1-x)$ where c is a constant. (Of course we don't need a computer to solve this equation, but by looking at different values of c we can see almost all the behaviors that come up in realistic fixed point problems.) If $1 < c \leq 4$, then this function maps the unit interval $[0, 1]$ into itself. Show that there are exactly two fixed points in the unit interval in this case. What are they exactly?

Answer: The equation $x = cx(1-x)$ is equivalent to $x[(1-c) - cx] = 0$, so there are exactly two solutions: $x = 0$ and $x = (1-c)/c$.

7. Use the graphical Matlab program `fixedpoint` to explore the behavior of fixed point iteration for this quadratic for various values of c in $(1, 4]$ and various initial values x_0. Determine as best you can for which values of c and x_0 the iteration converges to the positive fixed point. For which values does it converge to the fixed point at 0? For which values does it not converge to either? For which value of c is the convergence fastest? For which values is the convergence monotonic (i.e., increasing up to x^* or decreasing down to x^*).

Answer: For $1 < c \leq 3$ it converges. If $1 < c < 2$ the convergence is monotone; for $2 < c < 3$ the error alternates sign. For $c = 2$ the convergence is fastest. For $c > 3$ the iteration diverges. For larger values of c (c near 4), the behavior is very irregular (chaotic).

8. Suppose that $g(x^*) = x^*$ and that $|g'(x^*)| < 1$ (assume that g' exists and is continuous). Then x^* is called an "attractive fixed point." Prove that in this case there is a number $\epsilon > 0$ so that g is a contraction on the interval $[x^* - \epsilon, x^* + \epsilon]$. Conclude that if x^* is an attractive fixed point and x_0 is chosen sufficiently close to x^*, then the one point iteration converges to x^*.

Answer: Since $|g'(x^*)| < 1$ we can choose $\lambda < 1$ with $|g'(x^*)| < \lambda$. Since g' is continuous, there is a number $\epsilon > 0$ so that $|g'(x)| < \lambda$ for $x \in [x^* - \epsilon, x^* + \epsilon]$. For any $x, y \in [x^* - \epsilon, x^* + \epsilon]$ we have, by the Mean Value Theorem,

$$g(x) - g(y) = g'(\xi)(x - y),$$

with ξ between x and y. Therefore $|g'(\xi)| \leq \lambda$, so

$$|g(x) - g(y)| \leq \lambda |x - y|.$$

This shows that $g(x)$ is a contraction on the interval $[x^* - \epsilon, x^* + \epsilon]$. We also must show that g maps this interval into itself. If x is in this interval, then

$$|g(x) - x^*| = |g(x) - g(x^*)| \leq \lambda|x - x^*| \leq |x - x^*| \leq \epsilon,$$

so $g(x)$ is in the interval. Thus the previous problems imply that there is exactly one fixed point (namely x^*) in the interval, and that the one point interation converges to x^* starting from any x_0 in the interval.

9. If x^* is a fixed point but $|g'(x^*)| > 1$, then x^* is called a "repulsive fixed point." Show that for a repulsive fixed point there is a number $\epsilon > 0$ such that if x_n lies in the interval $[x^* - \epsilon, x^* + \epsilon]$ (but is not exactly the fixed point x^*), then x_{n+1} is actually further from x^* than x_n is. In other words, one point iteration does *not* converge to repulsive fixed points.

Answer: This is very similar to the last problem. Since $|g'(x^*)| > 1$ we can choose $\lambda > 1$ with $|g'(x^*)| > \lambda$. Since g' is continuous, there is a number $\epsilon > 0$ so that $|g'(x)| > \lambda$ for $x \in [x^* - \epsilon, x^* + \epsilon]$. If x_n lies in this interval, then (using the Mean Value Theorem)

$$|x_{n+1} - x^*| = |g(x_n) - g(x^*)| = |g'(\xi)||x_n - x^*|,$$

for some ξ between x_n and x^* (and so in the interval $x \in [x^* - \epsilon, x^* + \epsilon]$). Thus

$$|x_{n+1} - x^*| \geq \lambda|x_n - x^*|,$$

showing that the next iterate is actually further away.

10. Newton's method to find a root of $f(x)$ takes the form of an iteration $x_{n+1} = g(x_n)$. Show that the root of f is an attractive fixed point of g.

Answer: For Newton's method the iteration function is $g(x) = x - f(x)/f'(x)$. Therefore

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

Evaluated at a root x^* of f, we have

$$g'(x^*) = 0.$$

Thus, by problem 8, x^* is an attractive fixed point of $g(x)$.

11. If x_0 is sufficiently near a fixed point x^* of $g(x)$, it may happen that the further iterates (a) converge monotonically to x^*, (b) converge to x^* alternating between larger and smaller values than x^*, or (c) don't converge. Based on your work above, how can you predict which of these will occur in terms of the value of $g'(x^*)$?

Answer: As in previous problems we get

$$x_{n+1} - x^* = g(x_n) - g(x^*) = g'(\xi)(x_n - x^*).$$

If $0 < g'(x^*) < 1$, then the method will converge (see problem 8), and this formula shows that the error will not change sign from iteration to iteration, so the convergence is monotone; i.e., case (a) holds. If $-1 < g'(x^*) < 0$, the method converges, but the error switches sign each iteration; i.e., case (b) holds. If $|g'(x^*)| > 1$ then the iteration doesn't converge; case (c) holds.