Study Problems

1. a) Use an interpolating polynomial to find an approximation for $f'(0)$ in terms of $f(0)$, $f(h)$, and $f(2h)$.

b) Use Taylor's theorem to derive an error estimate for the approximation derived in part a).

2. a) Write an algorithm for which the input consists of the vectors d_i, l_i, u_i, and b_i, and the output is the vector x_i solving the tridiagonal system

$$\begin{pmatrix} d_1 & u_1 & 0 & 0 & \dots & 0 \\ l_1 & d_2 & u_2 & 0 & \dots & 0 \\ 0 & l_2 & d_3 & u_3 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & l_{n-2} & d_{n-1} & u_{n-1} \\ 0 & & \dots & & l_{n-1} & d_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

b) What is the operation count for your algorithm? Count multiplications and divisions as one operation each and ignore additions and subtractions.

3. If $\tilde{x}$ and $\tilde{y}$ are approximations of x and y with relative errors of at most 0.001 and 0.003, respectively, what can you say about the relative error of $\tilde{x}\tilde{y}$ as an approximation of xy?

4. Give an example where $\tilde{x}$ and $\tilde{y}$ are approximations of numbers x and y each with relative error of less than 1%, but for which $\tilde{x} + \tilde{y}$ has a relative error of more than 50% as an approximation of $x + y$.

5. a) Use the following table of data to approximate $\displaystyle\int_0^{1.5} \exp(-x^2)\,dx$ with Simpson's rule.

x	$\exp(-x^2)$
0.00	1.0000
0.25	0.9394
0.50	0.7788
0.75	0.5698
1.00	0.3679
1.25	0.2096
1.50	0.1054

b) Use Richardson extrapolation to estimate the error in your approximation.

6. Write an algorithm for which the input consists of the n-vector a_i and the scalar x and the output is

$$y = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n.$$

Use Horner's method.

7. Give an equation for the polynomial of lowest degree whose graph passes through the points $(1,0)$, $(-2,15)$, $(-1,0)$, and $(2,9)$.

8. If $e^{0.2}$ is estimated by interpolation from the values $e^0 = 1$, $e^{0.1} = 1.1052$, and $e^{0.3} = 1.3499$, is the error positive or negative? Find an upper bound and a lower bound on the error.

9. a) Write the simple trapezoidal rule approximation for $\int_0^h f(x)\,dx$.

b) Now divide the interval $[0,h]$ into 3 subintervals of length $h/3$ and write the composite trapezoidal rule approximation for this integral.

c) Show how to use Richardson extrapolation to combine the approximations of parts a) and b) to get a more accurate approximation.

10. Give the definition of the condition number of a nonsingular matrix A and use the condition number to describe the sensitivity of solutions of the linear systems $Ax = b$ to changes in b.

11. Find the LU decomposition of

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

12. Find the roots of the quadratic equation $x^2 - 40x + 1 = 0$ to five significant digits using $\sqrt{399} = 19.975$ (which is correct to five significant digits).

13. a) Determine a, b, and c so that $f(x)$ is a cubic spline:

$$f(x) = \begin{cases} 3 + x - 9x^3, & 0 \le x \le 1 \\ a + b(x-1) + c(x-1)^2, & 1 \le x \le 2 \end{cases}.$$

b) Is it a natural cubic spline?

14. Use Newton's method to derive an iteration which converges to $\sqrt[3]{2}$. Start with $x_0 = 1$ and compute two iterations.

15. Find weights w_i such that the rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(-\sqrt{3/5}) + w_2 f(0) + w_3 f(\sqrt{3/5})$$

is of as high a degree of precision as possible. What is the degree of precision?

16. Suppose that x^* is a fixed point of $g(x)$, that $g'(x^*) = 0$, and that $|g''(x)|$ is continuous and bounded. Starting from some x_0, define a sequence x_n by $x_{n+1} = g(x_n)$. Prove that

$$|e_{n+1}| \leq M|e_n|^2,$$

where $e_n = x_n - x^*$. Give an expression for M.

17. Let x_1, x_2, and x_3 be three distinct points and y_1, y_2, and y_3 three values. Let $p(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating the values y_1 and y_2 at the points x_1 and x_2, and let $q(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating y_2 and y_3 at x_2 and x_3. Prove that

$$r(x) = \frac{x_3 - x}{x_3 - x_1} p(x) + \frac{x - x_1}{x_3 - x_1} q(x)$$

is the Lagrange interpolating polynomial of degree ≤ 2 interpolating y_1, y_2, and y_3 at x_1, x_2, and x_3.

18. Write down the quadratic polynomial which interpolates the values of $\sqrt{x}$ at the points $x = 1$, 4, 9, using Newton's form of the interpolating polynomial.

19. Write down the formula for Newton's method applied to the equation $x^3 - \cos x = 0$.

20. Which of the following sequences appears to be converging to 1 quadratically?

a) 0.8027893, 0.9810498, 0.9980254, 0.9998018

b) 0.8027893, 0.9601194, 0.9984095, 0.9999975

c) 0.8027893, 0.9013942, 0.9500143, 0.9749988

d) 0.8027893, 0.9993521, 0.9992412, 0.9999999

21. Recall that IEEE single precision uses a 24 bit mantissa and an exponent running from -126 to 127. What is the smallest floating point number in this system which is greater than 10?

a) $10 + 10^{-7}$

b) $10 + 2^{-20}$

c) $10 + 2^{-23}$

d) $10 + 2^{-24}$

22. For each of the following statements, fill in the blank with one of the following: "the bisection method", "Newton's method", "the secant method", or "Muller's method".

a) ______________________ converges quickly if you start close to a root of the function, but it requires that you be able to compute the derivative of the function.

b) ______________________ requires that you know a bracketing interval.

c) ______________________ has an order of convergence of about 1.6.

d) ____________________ is based on quadratic interpolation.

e) The errors in ____________________ obey $|e_{n+1}| \approx M|e_n||e_{n-1}|$.

f) ____________________ may return a complex number as an approximation to the root, even if all the previous approximations are real numbers.

23. Find the Hermite cubic interpolant to $f(x) = \sqrt{x}$ at $x = 1$ and $x = 4$.

24. Consider the quadrature rule on $[-1, 1]$ given by

$$\int_{-1}^{1} f(x)\, dx \approx \frac{3}{4} f(-2/3) + \frac{1}{2} f(0) + \frac{3}{4} f(2/3).$$

a) What is the degree of precision of this rule?

b) Use scaling to write down the corresponding rule for an arbitrary interval $[a, b]$.

25. A certain cubic spline $s(x)$ for the knot sequence $x_0 = -2$, $x_1 = 0$, $x_2 = 1$, interpolates the data $y_0 = -7$, $y_1 = 1$, $y_2 = 4$ at the knots. If $s(x) = 2x^3 + 3x^2 + 2x + 1$ on the interval $[-2, 0]$, what is the formula for $s(x)$ on the interval $[0, 1]$?

26. When using the composite Simpson rule, halving the size of the subintervals typically reduces the error by about what factor?

a) 2 b) 4 c) 8 d) 16

27. Consider a cubic spline with knots $x_0 < x_1 < \cdots < x_n$ interpolating the values y_0, y_1, $\ldots$, y_n at the knots. Which of the following is true:

a) Given the choice of knots and values, there is a unique interpolating cubic spline.

b) Given the choice of knots and values, there is more than one interpolating cubic spline.

c) There is a unique interpolating cubic spline only if $n = 3$.

d) There may be an interpolating cubic spline or there may not be one, depending on the particular points and values.

28. Which of the following expressions *does NOT exactly equal* the error in the simple trapezoidal rule on $[a, b]$?

a) $\displaystyle\int_a^b f[a, b, x](x - a)(x - b)\, dx$

b) $\displaystyle -\frac{1}{12} f''(\eta)(b - a)^3$ for some $\eta \in [a, b]$

c) $\displaystyle -\frac{(b - a)^2}{12}[f'(b) - f'(a)]$

d) $\displaystyle\int_a^b \frac{(x - a)(x - b)}{2} f''(x)\, dx$