Study Problems

1. Find the polynomial of lowest degree whose graph passes through the points $(1,0)$, $(-2,15)$, $(-1,0)$, and $(2,9)$. Write the polynomial in three different ways: in Lagrange's form, in Newton's form, and in standard polynomial form.

2. Find the Cholesky decomposition of the matrix $\begin{pmatrix} 9/4 & -3 & 9/2 \\ -3 & 5 & -10 \\ 9/2 & -10 & 34 \end{pmatrix}$.

3. If $e^{0.2}$ is estimated by interpolation from the values $e^0 = 1$, $e^{0.1} = 1.1052$, and $e^{0.3} = 1.3499$, is the error positive or negative? Find an upper bound and a lower bound on the error.

4. Write down the algorithm to compute the Cholesky decomposition of a symmetric positive definite matrix. How many multiplications does it involve?

5. Precisely state the theorem giving the error in Lagrange interpolation.

6. Show that the following data can be interpolated by a polynomial of degree 3:

x :	-2	-1	0	1	2	3
$f(x)$:	1	4	11	16	13	-4

7. Give the definition of the condition number of a nonsingular matrix A and use the condition number to describe the sensitivity of solutions of the linear systems $Ax = b$ to changes in b.

8. Find the LU decomposition of $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.

9. a) Determine a, b, and c so that $f(x)$ is a cubic spline:

$$f(x) = \begin{cases} 3 + x - 9x^3, & 0 \le x \le 1 \\ a + b(x-1) + c(x-1)^2, & 1 \le x \le 2 \end{cases}.$$

b) Is it a natural cubic spline?

10. a) Under what conditions does a nonsingular matrix admit a decomposition as LU where L is unit lower triangular and U is upper triangular?

b) Under what conditions does a nonsingular matrix admit a decomposition as LDU where L is lower triangular, D is diagonal, and U is unit upper triangular.

c) Under what conditions does a nonsingular matrix admit a decomposition as $PA = LU$ where P is a permutation matrix, L is unit lower triangular, and U is upper triangular?

11. Supppose the function $\sin x$ is interpolated by a piecewise linear function using 101 equally spaced points from 0 to 5. Show that the maximum error in the interval $[0, 5]$ is no more than $1/800$. With a little more care, show the error is in fact bounded by $1/3,200$.

12. Supppose the function $\sin x$ is interpolated by a piecewise quadratic function using 101 equally spaced points from 0 to 5 and piecewise Lagrange interpolation. Show that the maximum error in the interval $[0, 5]$ is no more than $1/24,000$. Can you improve this bound?

13. Using a matrix norm of your choice, find the condition number of the matrix

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}.$$

State which norm you use.

14. Write down the quadratic polynomial which interpolates the values of $\sqrt[3]{x}$ at the points $x = 1, 8, 27$, using Newton's form of the interpolating polynomial.

15. Solve the system

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

using Gaussian elimination without pivoting.

16. Let x_1, x_2, and x_3 be three distinct points and y_1, y_2, and y_3 three values. Let $p(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating the values y_1 and y_2 at the points x_1 and x_2, and let $q(x)$ be the Lagrange interpolating polynomial of degree ≤ 1 interpolating y_2 and y_3 at x_2 and x_3. Prove that

$$r(x) = \frac{x_3 - x}{x_3 - x_1} p(x) + \frac{x - x_1}{x_3 - x_1} q(x)$$

is the Lagrange interpolating polynomial of degree ≤ 2 interpolating y_1, y_2, and y_3 at x_1, x_2, and x_3.

17. Find the LU decomposition of $\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

18. a) Give both Lagrange's and Newton's formula for the linear polynomial $p(x)$ interpolating $f(x)$ at the points $x = a$ and $x = b$.

b) Show that if f is C^2 on $[a, b]$ and $|f''| \leq M$ on $[a, b]$, then $|f(x) - p(x)| \leq M(b - a)^2/4$.

19. Find the Hermite cubic interpolant to $f(x) = \sin x$ at $x = 0$ and $x = \pi/4$.

20. A certain cubic spline $s(x)$ for the knot sequence $x_0 = -2$, $x_1 = 0$, $x_2 = 1$, interpolates the data $y_0 = -7$, $y_1 = 1$, $y_2 = 4$ at the knots. If $s(x) = 2x^3 + 3x^2 + 2x + 1$ on the interval $[-2, 0]$, what is the formula for $s(x)$ on the interval $[0, 1]$?

21. An upper Hessenberg matrix is an $n \times n$ matrix for which the entries $a_{ij} = 0$ whenever $i > j + 1$. In other words, a matrix for which the nonzero entries occurs in the positions marked with an $\times$:

$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times \\ \times & \times & \times & \dots & \times & \times \\ 0 & \times & \times & \dots & \times & \times \\ 0 & 0 & \times & \dots & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \times & \times \end{pmatrix}$$

a) Write an efficient algorithm for reducing an upper Hessenberg matrix to triangular form by Gaussian elimination without pivoting.

b) How many operations does your algorithm take (count multiplications and divisions as one operation and ignore additions)?

22. Consider a cubic spline with knots $x_0 < x_1 < \cdots < x_n$ interpolating the values y_0, y_1, $\dots$, y_n at the knots. Which of the following is true:

a) Given the choice of knots and values, there is a unique interpolating cubic spline.

b) Given the choice of knots and values, there is more than one interpolating cubic spline.

c) There is a unique interpolating cubic spline only if $n = 3$.

d) There may be an interpolating cubic spline or there may not be one, depending on the particular points and values.

23. If $s(x)$ is a cubic spline interpolating $f(x)$ at the knots x_0, x_1, $\dots$, x_n, then $s''(x)$ is:

a) a continuous piecewise linear function interpolating $f(x)$ at the knots.

b) a continuous piecewise linear function interpolating $f''(x)$ at the knots.

c) a continuous piecewise linear function which in general does not interpolate either $f(x)$ or $f''(x)$ at the knots

d) in general not a continuous piecewise linear function

24. Consider the decomposition of a matrix as LL^T where L is a nonsingular lower triangular matrix. Which of the following is true?

a) Any nonsingular matrix can be factored in this way.

b) Not every nonsingular matrix can be factored in this way, but only those that have nonzero diagonal matrices.

c) Not every nonsingular matrix can be factored in this way, but for every nonsingular matrix its rows may be permuted so that the resulting matrix can be factored in this way.

d) Only symmetric positive definite matrices can be factored in this way.