Study Problems

1. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0 + h/2] - f'(x_0)| \leq Nh^2.$$

Give a specific expression for N.

2. If $\tilde{x}$ and $\tilde{y}$ are approximations of positive numbers x and y with relative errors of at most 0.001 and 0.003, respectively, what can you say about the relative error of $\tilde{x} + \tilde{y}$ as an approximation of $x + y$?

3. Consider the simple quadrature rule on $[0, 1]$ derived as

$$\int_0^1 f(x)\,dx \approx \int_0^1 p(x)\,dx$$

where $p(x)$ is the linear polynomial interpolating $f(x)$ at $x = 0$ and $x = 3/4$.

a) Write down this quadrature rule in the usual form (a weighted sum of values of $f(x)$).

b) Scale this quadrature rule to obtain a simple quadrature rule approximation to $\int_a^b f(x)\,dx$, where $[a, b]$ is an arbitrary interval.

c) Write down the approximation to $\int_a^b f(x)\,dx$ obtained by applying this rule as a composite rule with n equal subintervals.

4. a) Obtain a quadrature rule on $[0, 1]$ as in the previous problem, but using the linear polynomial interpolating $f(x)$ at $x = 0$ and $x = c$, where, for the moment, c is an unspecified point in $[0, 1]$.

b) Is there any value of c for which this rule is exact for all quadratic polynomials $f(x)$?

c) Is there any value of c for which this rule is exact for all cubic polynomials $f(x)$?

d) For the best value of c (as determined in b and c above), write down the simple quadrature rule scaled to an arbitrary interval.

5. Find conditions on the number α which assure that the iteration $x_{n+1} = x_n + \alpha f(x_n)$ will converge to a zero of f for any starting value sufficiently near the root.

6. For what range of x is the approximation $\cos x \approx 1 - x^2/2$ correct to three decimal places rounded?

7. Consider the quadrature rule on $[-1, 1]$ given by

$$\int_{-1}^1 f(x)\,dx \approx \frac{3}{4}f(-2/3) + \frac{1}{2}f(0) + \frac{3}{4}f(2/3).$$

a) What is the degree of precision of this rule?

b) Use scaling to write down the corresponding rule for an arbitrary interval $[a, b]$.

8. Perform two iterations of Newton's method starting with $(x, y) = (0, 1)$ for the system

$$4x^2 - y^2 = 0, \qquad 4xy^2 - x = 1.$$

9. State and prove a theorem that says that, under appropriate hypotheses,

$$|f[x_0 - h/2, x_0, x_0 + h/2] - \frac{1}{2}f''(x_0)| \leq Ph^2.$$

Give a specific expression for P.

10. Devise a Newton iteration to compute the fifth root of a positive number.

11. Suppose the composite Simpson's rule with equally spaced points is used to approximate the integral of a smooth function. If the point spacing is decreased by a factor of 3, by what factor would you expect the error to decrease?

12. a) Use an interpolating polynomial to find an approximation for $f'(0)$ in terms of $f(0)$, $f(h)$, and $f(2h)$.

b) Use Taylor's theorem to derive an error estimate for the approximation derived in part a).

13. Use the following table of data to approximate $\int_0^{1.5} \exp(-x^2)\,dx$ with Simpson's rule.

x	$\exp(-x^2)$
0.00	1.0000
0.25	0.9394
0.50	0.7788
0.75	0.5698
1.00	0.3679
1.25	0.2096
1.50	0.1054

14. Use Newton's method to derive a quadratically convergent iteration to solve the *fixed point equation* $g(x) = x$.

15. Write the matrix $\begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$ as LU with L is lower triangular and U is unit upper triangular.

16. a) Write the simple trapezoidal rule approximation for $\int_0^h f(x)\,dx$.

b) Now divide the interval $[0, h]$ into 3 subintervals of length $h/3$ and write the composite trapezoidal rule approximation for this integral.

c) Show how to use Richardson extrapolation to combine the approximations of parts a) and b) to get a more accurate approximation.

17. Find weights w_i such that the rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(-\sqrt{3/5}) + w_2 f(0) + w_3 f(\sqrt{3/5})$$

is of as high a degree of precision as possible. What is the degree of precision?

18. Define $x_0 = -3$ and $x_{n+1} = x_n - \dfrac{x_n^2 - 4}{2x_n}$. Does $\lim_{n\to\infty}$ exists? If so what is the limit and what is the rate of convergence.

19. a) Find the interpolating polynomial of minimal degree for the data

x	0	2	3	4
y	7	11	28	63

b) By adding one extra term to $p(x)$, find the polynomial $q(x)$ which interpolates the above table and which satisfies $q(1) = 0$.

20. Write down the error expansion for the composite trapezoidal rule.

21. What is the smallest number > 70 in the IEEE single precision floating point arithmetic system.

22. Determine the natural cubic spline function interpolating the table

x	-1	0	1
y	13	7	9

23. The four point Lobatto quadrature rule, which has degree of precision 5, takes the form

$$\int_{-1}^{1} f(t)\,dt = w_1 f(-1) + w_2 f(\frac{-1}{\sqrt{5}}) + w_2 f(\frac{1}{\sqrt{5}}) + w_1 f(1).$$

a) What are the weights w_1 and w_2?

b) If you divide the interval $[0, 4]$ into four subintervals of length 1 and use the composite Lobatto rule to compute $\displaystyle\int_0^4 \exp(-x^2)\,dx$, how many evaluations of the exponential function are required?

c) If you then recompute using 8 subintervals of length 1/2, but about what factor will the error decrease?