Solutions for Homework 2

3.1 7. We know (see Theorem 2 in book), that $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0)$. Thus to guarantee that $|r - c_n| \le \epsilon$ we must choose n large enough that $2^{-(n+1)}(b_0 - a_0) < \epsilon$. Equivalently, $2^{-(n+1)} < \epsilon/(b_0 - a_0)$, or $-(n+1) < \log_2[\epsilon/(b_0 - a_0)]$, or $n + 1 > \log_2[(b_0 - a_0)/\epsilon]$, or, finally,

$$\boxed{n > \log_2 \frac{b_0 - a_0}{\epsilon} - 1}$$

9. (a) Since the width of the initial interval is $b_0 - a_0 = 2$, the width of the interval after n steps is $b_n - a_n = (b_0 - a_0)/2^n = \boxed{2^{1-n}}$.
(b) Since the root lies in the interval, it is at most half the length of the interval away from the midpoint of the interval. So it is at most $2^{1-n}/2 = \boxed{2^{-n}}$ away from the midpoint of the interval.
10. (a) true; (b) true (even if you replace n by $n + 1$); (c) COULD BE FALSE (e.g., $a_0 = -1$, $b_0 = 1$, $r = 0.1$. Then $a_1 = 0$, $b_1 = 1$, and $|r - 2^{-1}(a_1 + b_1)| > |r - 2^{-1}(a_0 + b_0)|$.); (d) true; (e) true; (f) COULD BE FALSE (this is the same as (c) above)
11. This is like 7 above, but now we want $|r - c_n|/|r| \le \epsilon$. Since $r \ge a_0 > 0$, and, as above, $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0)$, we have $|r - c_n|/|r| \le 2^{-(n+1)}(b_0 - a_0)/a_0$. With the same sort of algebra as in 7, we then get

$$\boxed{n > \log_2 \frac{b_0 - a_0}{a_0 \epsilon} - 1}$$

3.2 9. $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^3 - 2)/3x_n^2 = (2 + 2x_n^3)/(3x_n^3)$. Starting with $x_0 = 1$ we get $x_1 = 4/3$, $\boxed{x_2 = 91/72}$.
10. We use Newton's method to find a root of $f(x) = x^3 - R$. This gives $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^3 - R)/3x_n^2 = (R + 2x_n^3)/(3x_n^3)$. A glance at the graph of $y = x^3 - R$ (which is the graph of $y = x^3$ translated down a distance R), shows that if $x_0 > 0$, then $x_1 \ge x_2 \ge \ldots \ge \sqrt[3]{R}$ (since $x^3 - R$ is strictly increasing and convex for $x > 0$). Thus *Newton's method will converge for any* $x_0 > 0$. (For $x_0 \le 0$ things are less predictable. For $x_0 = 0$ the method breaks down, because the computation of x_1 would involve division by 0. Then there is another value of x_0 that leads to $x_1 = 0$ and thus breakdown again. Continuing, in this way we see that there are (infinitely many) values of $x_0 < 0$ for which the method breaks down.)

23. (a) For this example we have $f_1(x_1,x_2) = 4x_1^2 - x_2^2$, $f_2(x_1,x_2) = 4x_1x_2^2 - x_1 - 1$. Therefore the Jacobian matrix is

$$J = \begin{pmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1x_2 \end{pmatrix}.$$

Starting with $\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get for the first iteration

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}, \quad J^{-1} = \begin{pmatrix} 0 & 1/3 \\ -1/2 & 0 \end{pmatrix},$$
$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -J^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/2 \end{pmatrix}, \quad \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix},$$

and for the second iteration

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 7/36 \\ -1 \end{pmatrix}, \quad J = \begin{pmatrix} 8/3 & -1 \\ 0 & 4/3 \end{pmatrix}, \quad J^{-1} = \begin{pmatrix} 3/8 & 9/32 \\ 0 & 3/4 \end{pmatrix},$$
$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -J^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 5/24 \\ 3/4 \end{pmatrix}, \quad \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 13/24 \\ 5/4 \end{pmatrix}.$$

(b) In this example the starting point $(1,1)$ happens to be a root, so iterating doesn't change it. (I wonder if the authors meant this to be, or it was an oversight.)

3.3 2.

4. If $\lim_{n\to\infty} x_n = q$, then $\lim_{n\to\infty} x_{n+1} = q$ and $\lim_{n\to\infty} x_{n-1} = q$ as well. Since $f(x)$ is continuous, $\lim_{n\to\infty} f(x_n) = f(q)$. Next, consider the limit of the quantity

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

By the Mean Value Theorem this equals $f'(\xi_n)$ for some ξ_n between x_n and x_{n-1}. Since both x_n and x_{n-1} converge to q, the same is true of ξ_n, and so $f'(\xi_n)$ converges to $f'(q)$ (assuming that f is C^1). Thus we have shown that

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = f'(q).$$

Now the definition of the secant method is

$$x_{n+1} = x_n - f(x_n)\left[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right]^{-1}.$$

Using the above calculations we can take limits on both sides to get

$$q = q - f(q)[f'(q)]^{-1}$$

(here we use that $f'(q) \neq 0$. Solving this equation for $f(q)$ gives $f(q) = 0$.

7. This is straightforward algebra:

$$\begin{aligned} x_{n+1} &= x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{f(x_n)x_n - f(x_{n-1})x_n - f(x_n)x_n + f(x_n)x_{n-1}}{f(x_n) - f(x_{n-1})} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}. \end{aligned}$$

This is an inferior formula, because as x_n converges to a root, both the numerator and the denominator involve the subtraction of two nearly equal quantities, and thus can lead to loss of significance.