CSE/MATH 455, Arnold Lab 2 September 19, 1997

Stability, cancellation, and conditioning problems with polynomials

Preparation: For this lab you need to copy the file quadroots.m from my directory into yours. Use the Unix command "`cp dna2/quadroots.m .`" (don't forget the period which is an abbreviation for your current directory).

1. Use Matlab to plot the polynomial

$$f(x) = (x-1)^7 = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1,$$

on the interval $[0.988, 1.012]$. Evaluate the function in three different ways: (1) using the factored form above, (2) using the expanded forms above, and (3) using nested multiplication (Horner's method): $f(x) = ((((((x-7)x+21)x-35)x+\cdots)x-1$.

Plot the three graphs on top of each other in different colors so you can see the differences. The point is that, although the results would be the same in exact arithmetic, the computed results are quite different. Let's examine why the expanded form is so bad.

To investigate consider the value $x = 1.01$. Compute the sum of all terms of the expanded form except for the final subtraction of 1. That is, what is

$$x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x?$$

when $x = 1.01$? *Don't* use the computer for this, just do $(1.01-1)^7 + 1$ by hand.
Answer: _____

Now when you compute the final subtraction of 1 from this number how many decimal places of accuracy can you expect to lose? _____

2. In this problem we will consider the effect of computer arithmetic on the evaluation of the quadratic formula

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of $p(x) = ax^2 + bx + c$.

The simple Matlab function `quadroots` provided applies this formula. If you give it a vector of length three with the coefficients a, b, and c, `quadroots` will return a vector of length 2 with the computed roots. Look over `quadroots` and try it on a few quadratics for which you know the answers to be sure you know how to use it and to be sure it works.

The product of the roots of $ax^2 + bx + c$ is of course c/a. This gives us a way to check `quadroots`. The statement `( p(3)/p(1) - prod(quadroots(p)) ) / ( p(3)/p(1) )` shows the relative error in this product (it should always give zero).

Test `quadroots` in this way for the following polynomials:

$2x^2 + 6x - 3$
$x^2 - 14x + 49$
$3x^2 - 123454321x + 2$

Why are the results so bad for the last polynomial? For this polynomial what are the roots reported by `quadroots`? (Use `r=quadroots(p)` and then print `r(1)` and `r(2)` separately so you can see the significant figures.)

Larger root: _____ Smaller root: _____

The correct answers, rounded to six decimal places, are

Larger root: 4.11514×10^7 Smaller root: 1.62003×10^{-8}

(Matlab has a command `roots(p)` which returns the roots of the polynomial with coefficients in `p`. `roots` is smarter than `quadroots` and will give accurate answers here.) What were the relative errors of the answers returned by `quadroots`?

Larger root: _____ Smaller root: _____

Which root is better? Why?

Use the fact that the product of the roots is c/a to compute the bad root more accurately.

A useful (optional) exercise is to rewrite quadroots so that it gives accurate answers even when $|b^2|$ is much greater than $|4ac|$.

3. Wilkinson's example demonstrates an entirely different problem with computing the roots of polynomials. Give the command `p=poly(1:20)`. This gives the vector of coefficients for the polynomial with roots $1, 2, \ldots, 20$. If you give the command `roots(p)` you should get back the roots, $1, 2, \ldots, 20$. Actually most of the roots are quite accurate, but some have errors as large as 0.25%. This is a tough problem and even a good routine has difficulties giving accurate answers.

Now change the first coefficient from 1 to 1.00001: `p(1)=1.00001` and run `roots(p)` again. Notice what kind of change takes place in the roots. This is Wilkinson's famous example.

Note this is not the fault of `roots`, but due to the fact that the problem of finding the roots of this polynomial is very badly conditioned. (Reread the explanation in the book again.)

4. Another example of bad conditioning in finding roots of polynomials occurs when there are multiple roots. For example consider the polynomial $x^3 - 6x^2 + 12x - 8$ which has a triple root at $x = 2$. Study the effect of perturbations of the coefficients on this root.