CMPSC/MATH 455, Arnold Lab 4 October 2, 1997

Fixed points and fixed point iteration

A quick review of some concepts from Wednesday's class. A number x^* is a fixed point of a continuous function $g(x)$ if $g(x^*) = x^*$. A continuous function which maps a closed bounded interval into itself always has a fixed point. Fixed point iteration is a simple algorithm for computing fixed points: pick x_0 and define $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$.

Copy the file `~dna2/fixedpoint.m` to your own directory. Use a separate sheet of paper to record the answers to the following questions.

1. We are going to explore fixed point iteration for quadratic functions of the form $g(x) = cx(1-x)$ where $c > 0$ is a constant, i.e., functions whose graphs are parabolas crossing the x-axis at $x = 0$ and $x = 1$. Of course we don't need a computer to solve this equation, but by looking at different values of c we can see almost all the behaviors that come up in realistic fixed point problems. If $0 \leq c \leq 4$, then this quadratic maps the unit interval $[0, 1]$ into itself. Show that if $0 \leq c \leq 1$ then $x = 0$ is the only fixed point of g on the unit interval, but that if $1 < c \leq 4$ there are two fixed points in the unit interval. Give a formula for the values of the two fixed points in this case.

2. The Matlab program `fixedpoint` is designed to explore the behavior of fixed point iteration for this quadratic for various values of c in $[0, 4]$ and various initial values x_0. Just start Matlab and type `fixedpoint` to invoke it. Use `fixedpoint` to answer the following questions as completely as you can. Start with $c = 1$, 2, 3, and 4, and then try 0.5, 1.5, 2.5, and 3.5, and then more values as needed to see what's going on. For each value of c try a couple of values of x_0.

   a. For which values of c and x_0 does the iteration converges to the positive fixed point?

   b. For which values does it converge to the fixed point at 0?

   c. For which values does it not converge to either?

   d. For which value of c is the convergence fastest?

   e. For which values is the convergence monotonic (i.e., increasing up to x^* or decreasing down to x^*).

3. Suppose that $g(x^*) = x^*$ and that $|g'(x^*)| < 1$ for some C^1 function g. Then x^* is called an "attractive fixed point." As discussed in class, if x^* is an attractive fixed point

and x_0 is chosen sufficiently close to x^*, then the one point iteration converges to x^*. If x^* is a fixed point but $|g'(x^*)| > 1$, then x^* is called a "repulsive fixed point." In this case, when an iterate x_n is sufficiently close to the fixed point, the next iterate x_{n+1} will actually be further away. Consequently, fixed point iteration does *not* converge to repulsive fixed points.

a. For which values of c between 0 and 4 is the fixed point of $g(x) = cx(1-x)$ at the origin attractive? For which values is it repulsive?

b. For which values of c between 1 and 4 is the other fixed point attractive? For which values is it repulsive?

c. How do these results match up with the observations from the previous problem?

d. What do you observe when both fixed points are repulsive?

4. Notice that for $c = 2$ the convergence to the fixed point at $x = 1/2$ is quite fast, even if you start near the fixed point at $x = 0$. For $c = 1$ or $c = 3$ the convergence is very slow. Relate this to $g'(x^*)$.

5. If x_0 is sufficiently near a fixed point x^* of $g(x)$, it may happen that the further iterates (a) converge monotonically to x^*, (b) converge to x^* alternating between larger and smaller values than x^*, or (c) don't converge. Based on your work above, how can you predict which of these will occur in terms of the value of $g'(x^*)$?

6. Newton's method to find a root of $f(x)$ takes the form of an iteration $x_{n+1} = g(x_n)$. For example, if $f(x) = x + \sin 3x - 2$, then Newton's method to find a root of $f(x)$ is given by $x_{n+1} = x_n - (1 + 3\cos 3x_n)/(x_n + \sin 3x_n - 2)$, which is exactly the same as fixed point iteration to find a fixed point of $g(x) = x - (1 + 3\cos 3x)/(x + \sin 3x - 2)$. For Newton's method applied to a general function $f(x)$, what is the formula for the corresponding function $g(x)$? Show that if x^* is a simple root of f, then x^* is an attractive fixed point of g. Compute $g'(x^*)$ in this case. What does this have to do with the fast convergence of Newton's method?

7. Optional problem (a good way to check your understanding). Look up Steffenson's method in the book. Write it as a fixed point iteration and compute $g'(x^*)$ for this iteration.