CSE/MATH 455, Arnold | Lab 8 | November 6, 1997

The Chebyshev Points

Hand in this lab at the end of class, if finished, or by Wednesday 11/12. Preparation: For this lab you need to copy the following files from `~dna2` into your working directory: eprod.m, chebpts.m. If you don't have ninterp.m from last time, copy that too.

Recall that if $p(x)$ is the Lagrange interpolating polynomial which interpolates the values of a function $f(x)$ at the $n+1$ points $x_0, x_1, \ldots, x_n$, then the error is given by the formula

$$f(t) - p(t) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (t - x_i),$$

where ξ is some point depending on the x. Suppose we want to make this error as small as possible for a given number of n points and a given function $f(x)$. We don't have much control over the first two terms: $1/(n+1)!$ and $f^{(n+1)}(\xi)$, but if we can choose the interpolation points x_i, we can use this to adjust the size of the last term $\prod_{i=0}^{n}(t - x_i)$. This product, which gives a polynomial of degree $n+1$ with roots at $x_0, x_1, \ldots, x_n$, is called the *error product*.

If we are only interested in the error at one point x, we can easily make the error zero there: just take $x_0 = x$. But if we want to make the error small on a whole interval I, then we are lead to the question: *what choice of the points $x_0, x_1, \ldots, x_n$ makes the maximum absolute value of the error product,* $\max_{t \in I} \left| \prod_{i=0}^{n} (t - x_i) \right|$, *as small as possible?*

1. First experiment with just three points. For simplicity take the interval $I = [-1, 1]$. Start with three evenly spaced points including the end points: set `x=[-1,0,1]`. Next set `t=linspace(-1,1,200);` to generate 200 points from -1 to 1, and set `y=eprod(x,t);`. This evaluates the product $\prod(t - x_i)$ at the 200 points. Make a plot with `plot(t,y)`. Examine the graph to estimate maximum absolute value. ______________ (It helps to type `grid`.) Use `max(abs(y))` to get the maximum more accurately: ______________ .

2. Now try moving the endpoints in. Change `x` to `[-.9,0,.9]` and plot the product again, and evaluate the maximum. What is it now? ______________ Did it go up or down? ______________

3. Keep moving the endpoints around and evaluating $\max_{t \in I} |\prod_{i=0}^{n}(t - x_i)|$. What choices for the points x_i gives the smallest value (try to get the points correct to two decimal places)? ______________ , ______________ , ______________

4. It turns out that for any interval $I = [a, b]$ and any number of points n, there is a formula for the $n+1$ points $x_0, x_1, \ldots, x_n$ which makes the maximum of the absolute value of the error product on I smallest. The formula is $x_i = \frac{a+b}{2} + \frac{a-b}{2} \cos\left(\frac{(2i+1)}{2n+2}\pi\right)$, $i = 0, 1, \ldots, n$. These are the famous *Chebyshev points*, named after the great mathematician who discovered them, Pafnuty Lvovitch Chebyshev (1821–1894) (a.k.a. Pafnutii L'vovitsch Tschebysheff, Paphnutij L'vovič Cebysev, and by a few hundred other spellings). The matlab command `chebpts(a,b,n+1)` returns a vector of length $n+1$ containing these points. Use it to find the true best values of x_0, x_1, and x_2 from the last problem. ______________, ______________, ______________ Were you right?

5. Now use 10 evenly spaced points from -1 to 1 (`x=linspace(-1,1,10)`). What is the maximum absolute value of the error product? ______________ Look at the plot. Does the maximum error occur near the middle of the interval, near the ends, equally at both places, or what? ____________________________

6. Now answer the same questions but using the 10 Chebyshev points. Max? __________ Where? ____________________________

7. What is the max absolute value of the error product for 30 equally spaced points ______________ and 30 Cheybshev points ______________ ? Also, look at the Chebyshev points with `plot(x,0*x,'o')` (where `x` holds the Chebyshev points). Where are the points most closely spaced? ____________________________

8. Now let's see how much difference it can make to interpolate at the Chebyshev points. We'll use Runge's example from the last lab: interpolating the function $f(x) = 1/(1+x^2)$ on the interval $I = [-5, 5]$. First use `ninterp` to evaluate the interpolating polynomial using 10 equally spaced points from -5 to 5 (if you don't remember how to use `ninterp`, do `help ninterp`). Plot the result and compare it to the true function f. Roughly what is the maximum error? ______________ Now use 10 Chebyshev points and do the same. Roughly what is the maximum error now? ______________ . How about with 30 points? Evenly spaced: ______________ Chebyshev: ______________

9. (Optional extra credit problem). Investigate the approximation to $f(x) = |x|$ on $I = [-1, 1]$ using equally spaced points and Chebyshev points. Report your results on a separate sheet. (A question to investigate is: at what points do the interpolating polynomials converge as you increase the number of equally spaced or Chebyshev points?)