MATH 502 Second exam May 9, 1997 Professor Arnold

Do 3 problems, omit 1. State which problem you omit. Partial credit will be awarded, but errors and unacknowledged omissions count negatively.

1. Suppose that $x, x_1, x_2, \ldots$ are elements in a Hilbert space such that x_n converges to x weakly and $\|x_n\|$ converges to $\|x\|$. Prove that x_n converges to x in norm. Give a counterexample in Banach space.

Under the given hypotheses,

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\langle x_n, x\rangle = \|x_n\|^2 - \|x\|^2 - 2\langle x_n - x, x\rangle \to 0.$$

For a counterexample in Banach space, consider $x_n = e_1 + e_n$ in c_0, where the e_n are the usual sequence space basis elements. Then $\|x_n\| = 1$ for all $n > 1$ and $x_n \to e_1$ weakly.

2. Let X and Y be Banach spaces and set

$$U = \{\, T \in B(X,Y) \,|\, T^*(Y^*) = X^* \,\}.$$

Prove that U is open in $B(X,Y)$.

Since the null-space of T is the annihilator of range of T^*, T is one-to-one iff T^* has dense range. Combining with the closed range theorem, we get that T is one-to-one with closed range iff T^* has dense, closed range, i.e., iff T^* is surjective. Thus U is the set of operators which are one-to-one with closed range, which is the set of operators for which there exists a $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all x. If $T \in U$ and $c > 0$ is the corresponding constant, we show that $T + S \in U$ for all $\|S\| \leq c/2$. Indeed

$$\|(T+S)x\| \geq \|Tx\| - \|Sx\| \geq c\|x\| - c/2\|x\| = c/2\|x\|.$$

3. Prove that the sum of a closed subspace and a finite dimensional subspace of a Banach space is closed.

It is enough to consider a closed space S and one element $x \notin S$ and show that $S + \mathbb{R}x$ is closed. By the Hahn-Banach Theorem there is a bounded linear functional f which vanishes on S and satisfies $f(x) = 1$. Now suppose that $y_n \in S + \mathbb{R}x$ and $y_n \to y$. Say $y_n = s_n + r_n x$, $s_n \in S$, $r_n \in \mathbb{R}$. Then $r_n = f(y_n) \to f(y)$. Also $s_n = y_n - r_n x \to y - f(y)x$, and, since S is closed, $y - f(y)x \in S$. Thus $y = [y - f(y)x] + f(y)x \in S + \mathbb{R}x$.

Alternate proof: $S + F = \pi^{-1}(\pi F)$ where π is the projection of the Banach space X onto X/S. Since F is finite dimensional, πF is finite dimensional, hence closed in X/S, and its inverse image under a continuous map is closed as well.

4. Prove that any compact subset of the complex plane is the spectrum of a bounded linear operator on a Banach algebra.

Let K be the compact set and let X be the space of continuous complex valued functions on K. Define $T : X \to X$ by $Tf(z) = zf(z)$. If $\lambda \notin K$, then $\mathrm{dist}(\lambda, K) > 0$, so $1/(z-\lambda)$ is bounded on K, so multliplication by $1/(z-\lambda)$ is continuous on K, and provides the inverse of $T - \lambda\mathbf{1}$. Thus $\lambda \notin \sigma(T)$. Conversely, if $\lambda \in K$, then we can take $f \in X$ of norm one with $(T-\lambda\mathbf{1})f$ arbitrarily small (e.g., if f is chosen to takes values in $[0,1]$ and vanish when $|z-\lambda| \geq \epsilon$, then $\|(T-\lambda I)f\| \leq \epsilon$). This shows that $T - \lambda\mathbf{1}$ cannot be invertible; i.e., $\lambda \in \sigma(T)$. Note: there are variants of this construction. E.g., we could choose a dense sequence of numbers in K and consider the corresponding diagonal matrix on ℓ_2.