MATH 502, Spring '97 Homework 10 Arnold

1. Prove that a Banach space is reflexive if and only if its dual space is reflexive.

2. Let X be a Banach space and S a closed subspace. Prove that X is reflexive if and only if S and X/S are reflexive.

3. Let X be a Banach space, $j : X \to X^{**}$ the Riesz map, and V a closed subspace of X^*. Show that $j({}^aV) = V^a \cap \mathcal{R}(j)$.

4. Let X be a reflexive Banach space and S a closed subspace of X, and V a closed subspace of X^*. Show that $j(S) = S^{aa}$ and that $({}^aV)^a = V$.

5. Suppose that a Banach space X is the direct sum of two closed subspaces M and N (i.e., every element of X has a unique decomposition as $m + n$ with $m \in M$, $n \in N$). Define $P : X \to M$ by $P(m+n) = m$. Prove that P is a projection operator, i.e., that it is a bounded linear idempotent operator (idempotent means $P^2 = P$), and that $\mathcal{R}(P) = M$, $\mathcal{N}(P) = N$. Conversely, prove that if Q is a projection operator on X, then X is the direct sum of the closed subspaces $\mathcal{R}(Q)$ and $\mathcal{N}(Q)$.

6. Suppose that the Banach space X admits a Schauder basis $\{x_n\}_{n=0}^{\infty}$. For fixed $k \in \mathbb{N}$, define $f : X \to \mathbb{R}$ by $f(\sum_{n=0}^{\infty} a_n x_n) = a_k$. Prove that $f \in X^*$.