1. Given an example of a Banach space X and a proper closed subspace S of X^*, for which S is weak* dense in X^*.

2. Prove that if S is a finite dimensional subspace of X^* then $({}^aS)^a = S$.

Bilinear Forms

Many important problems, e.g., boundary value problems for partial differential equations, can be expressed as follows.

Problem B. Given Banach spaces X and Y, a mapping $B : X \times Y \to R$, and $f \in Y^*$, find $x \in X$ such that

$$B(x, y) = f(y) \qquad \forall y \in Y.$$

3. Let X and Y be Banach spaces. The Cartesian product $X \times Y$ is a Banach space with the norm $\|(x, y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$. A functional $B : X \times Y \to \mathbb{R}$ is bilinear if $B(x, \cdot)$ is linear on Y for each fixed x and $B(\cdot, y)$ is linear on X for each fixed y. Prove that the following statements about a bilinear functional are equivalent:

1. B is continuous on $X \times Y$.
2. There exists c such that $|B(x, y)| \le c\|x\|\|y\|$ for all $x \in X$, $y \in Y$.
3. B is separately continuous (i.e., continuous in x for fixed y and vice versa).

4. Define a norm on the space of continuous bilinear functionals $\mathrm{Bil}(X, Y)$ and prove that it is a Banach space.

5. Show how to associate to each $B \in \mathrm{Bil}(X, Y)$ maps $S \in B(X, Y^*)$ and $T \in B(Y, X^*)$. Show that if X and Y are reflexive, then $S^* = T$ and $T^* = S$.

6. Assume that X and Y are reflexive. Show that the *inf-sup condition*

$$\gamma := \inf_{0\neq x\in X} \sup_{0\neq y\in Y} \frac{B(x, y)}{\|x\|\|y\|} > 0$$

holds if and only if S is one-to-one with closed range. The bilinear form is said to be *proper* if the inf-sup condition holds, and for all $y \in Y$ there exists $x \in X$ with $B(x, y) \neq 0$. Prove that the bilinear form is proper if and only if S is an isomorphism of X onto Y^*, and that this holds if and only if the Problem B has a unique solution for each $f \in Y^*$.

7. Let B be a proper bilinear form on reflexive Banach spaces and define the inf-sup constant γ as above. Prove that

$$\gamma = \inf_{0\neq y\in Y} \sup_{0\neq x\in X} \frac{B(x, y)}{\|x\|\|y\|}.$$

8. Prove the Lax-Milgram lemma: If $B : X \times X \to \mathbb{R}$ is a bilinear form on a reflexive Banach space which is positive in the sense that for some $c > 0$, $|B(x, x)| \ge c\|x\|^2$ for all $x \in X$, then for all $f \in X^*$ there exists a unique $x \in X$ such that $B(x, y) = f(y) \quad \forall y \in X$.