For $T_n, T \in B(X,Y)$ we say that T_n converges to T in the strong operator topology if $T_n x$ converges to Tx in Y for all x in X. We say that T_n converges to T in the weak operator topology if $f(T_n x)$ converges to $f(Tx)$ for all $x \in X$, $f \in Y^*$. Note that both these topologies reduce to the weak* topology in case $Y = \mathbb{R}$, but in general they are different topologies and are both weaker than the norm topology.

1. Given an example of a sequence of operators T_n from a Hilbert space to itself which converge to some operator T in the weak operator topology, but not in the strong operator topology.

2. Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Show that $x_n \in H$ converges weakly to $x \in H$ if and only if

$$\sup_n \|x_n\| < \infty \qquad \text{and} \qquad \lim_{n\to\infty} \langle x_n, e_m \rangle = \langle x, e_m \rangle \quad \forall m.$$

3. With the same notation as in the previous problem, let $T \in B(H)$. Define an infinite matrix $(k_{ij})_{i,j=0}^{\infty}$ by $k_{ij} = \langle Te_j, e_i \rangle$. Show that if T_n is a norm bounded sequence in $B(H)$ with matrix $k_{ij}^{(n)}$, then (a) T_n converges to T in the weak operator topology iff the matrix elements converge $k_{ij}^{(n)} \to k_{ij}$ $\forall i,j$; (b) T_n converges to T in the strong operator topology iff the column vectors $(k_{ij}^{(n)})_{i=0}^{\infty}$ converge to the corresponding column vector $(k_{ij})_{i=0}^{\infty}$ in l_2 as $n \to \infty$.

4. Let X be an infinite dimensional Banach space. Show that X contains a subspace of finite codimension which is not closed.

5. Let f_i be a sequence of continuous real-valued functions on $[0,\pi]$, all bounded by 1. Prove that subsequence f_{i_k} exists such that for every integer m the limit

$$\lim_{k\to\infty} \int_0^{\pi} f_{i_k}(x) \sin mx \, dx$$

exists.

6. Let S and T be linear transformations from a Hilbert space H to itself such that $\langle Sx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. Prove that S and T are bounded operators.