MATH 502, Spring '97 Homework 13 Arnold

1. Prove that the inclusion operator of $C^1([0,1])$ into $C([0,1])$ is compact.

2. Let X and Y denote Banach spaces and suppose that that $x_n \to \bar{x}$ weakly in X. Prove that if $T \in B(X,Y)$, then $Tx_n \to T\bar{x}$ weakly and if $T \in B_c(X,Y)$, then $Tx_n \to T\bar{x}$ in norm.

3. Give an example of a sequence of closed bounded subsets of a Banach space with $B_1 \supset B_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} B_i = \emptyset$. On the other hand, prove that if the B_i are also required to be closed balls, no such example exists.

4. For $s \in \mathbb{R}$ define an operator $T_s : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T_s f(x) = f(x+s)$. Show that for every $f \in L^2(\mathbb{R})$, the map $s \mapsto T_s f$ is a continuous from $\mathbb{R}$ to $L^2(\mathbb{R})$. Show that the map $s \mapsto T_s$ is *not* continuous from $\mathbb{R}$ to $B(L^2(\mathbb{R}), L^2(\mathbb{R}))$.

5. Let H be a complex Hilbert space and f an *unbounded* linear funtional on H. Prove that $\mathcal{N}(f)$ is dense in H, but not closed in H.

6. Prove that the weak closure of the unit sphere in l^2 is the unit ball in l^2.