MATH 502, Spring '97 Homework 9 Arnold

1. Prove that if X^* is separable for some Banach space then X is as well. Is the converse true (X separable $\implies$ X^* separable)?

2. Let X be a Banach algebra with an identity (i.e., X is a Banach space that also has a continuous multiplication which makes it a (possibly non-commutative) algebra. Assume that the identity element, which we denote by 1, has norm equal to the real number 1. Prove that if x belongs to the open unit ball then $1 - x$ is invertible.

3. Under the assumptions of 3, show that the set of invertible elements of X is open.

4. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, both endowed with the l_1 norm. Let $A : X \to Y$ be a linear operator with matrix (a_{ij}). Give an explicit expression of $\|A\|_{B(X,Y)}$ in terms of the matrix entries. Do the same if the Euclidean spaces are endowed with the l_∞ norms.

5. Let X be a normed vector space. Prove that every continuous additive functional $f : X \to \mathbb{R}$ belongs to X^*. (Additive means that $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.)

6. Let S be a closed subspace of a Hilbert space and f a linear function on S with norm 1. Prove that f has a *unique* extension to the whole space maintaining norm 1. Is the same result true in Banach space?