MATH 502 Midterm March 7, 1997 Professor Arnold

Do 3 problems, omit 1. State which problem you omit. Partial credit will be awarded, but errors and unacknowledged omissions count negatively.

1. Prove or provide a counterexample (with justification): if f is analytic on $B(0,1)$ and continuous on its closure, then f extends to an analytic function on $B(0,R)$ for some $R > 1$.

It's false. One counterexample is given by $f(z) = \sum_{n=0}^{\infty} z^n/n^2$. *The radius of convergence is given by*

$$\frac{1}{R} = \limsup_{n\to\infty} \frac{1}{n^{2/n}} = \limsup e^{-2\,n\log n} = 1,$$

so this function is not analytic on any disk strictly containing $\bar{B}(0,1)$. *However* $\sum 1/n^2 < \infty$, *so the Weierstrass M-test implies that the series converges absolutely and uniformly on the closed disk* $\bar{B}(0,1)$, *and hence* f *is continuous there. Another counterexample is simply* $f(z) = \sqrt{z-1}$.

2. (a) Suppose that f is analytic and nowhere zero on some bounded region of the complex plane, continuous on its closure, and of constant modulus on its boundary. Prove that f is constant on the region.

(b) Can the hypothesis that f is nowhere zero be dropped?

(c) Can we replace the hypothesis that the region is bounded by that its complement is unbounded?

(a) Let c *denote the constant value of* $|f|$ *on the boundary. Then, by the maximum modulus principle,* $|f| \le c$ *on the region and* f *is constant if equality holds at any point. If* $c = 0$ *this shows that* $f \equiv 0$ *which is a contradiction to the hypotheses. Thus* $c > 0$ *and, since* f *is nowhere zero, the function* $1/f$ *is analytic, continuous on the closure, and of constant modulus* $1/c$ *on the boundary. Thus* $|1/f| \le 1/c$, *or* $|f| \ge c$. *Thus* $|f| = c$, *and so* f *is constant.*

(b) No. Let $f(z) = z$ *on the unit disk.*

(c) No. Let $f(z) = e^z$ *on the right half plane.*

3. How many roots (counting multiplicity) does the function $p(z) = z^6 + z^3 + 5z^2 - 2$ have in the annulus $1 < |z| < 2$?

On the circle $|z| = 1$, $|5z^2| = 5$ *and* $|z^6 + z^3 - 2| \le 4$, *so that* $p(z)$ *has no zero on* $z = 1$ *and, by Rouché's theorem, it has the same number of zeros as* $5z^2$ *inside the circle, namely 2. On* $|z| = 2$, $|z^6| = 64$ *and* $|z^3 + 5z^2 - 2| \le 30$, *so again* p *has no root on the circle and Rouché's theorem implies that it has 6 roots inside the circle. Thus* p *has* $4 = 6 - 2$ *roots in the annulus.*

4. Find $\int_0^{2\pi} \sin(\exp e^{i\theta})\, d\theta$.

If $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ *or* $d\theta = -i\,dz/z$, *so the integral is* $-i\int_\gamma \sin(\exp z)/z\,dz$ *where* γ *is the unit circle. The integrand is analytic except for the pole at zero, where the residue is* $\sin(\exp 0) = \sin 1$. *By the Residue Theorem the integral is* $-i \cdot 2\pi i \sin 1 = 2\pi \sin 1$.