1. The 3-point difference operator is $D_h^2 u(x) = \dfrac{u(x-h) - 2u(x) + u(x+h)}{h^2}$ and it satisifes

$$|D_h^2 u(x) - u''(x)| \le \frac{h^2}{12} M_4, \tag{1}$$

where M_4 is the maximum of $|u''''|$ on the interval $(x-h, x+h)$. Find the best possible approximation of $u''(x)$ using *five* values, $u(x-2h), u(x-h), \ldots, u(x+2h)$, and state and prove an error equation for it analogous to (1).

2. In this problem I use *stencil notation* to describe a finite difference scheme. For example, the usual 5-point Laplacian has the stencil shown on the left of this display:

$$\frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} & & \alpha & & \\ & & \beta & & \\ \alpha & \beta & \gamma & \beta & \alpha \\ & & \beta & & \\ & & \alpha & & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} \alpha & \beta & \alpha \\ \beta & \gamma & \beta \\ \alpha & \beta & \alpha \end{pmatrix}$$

a) Consider a 9-point difference approximation to the Laplacian with stencil as given above in the middle. Is it possible to choose the constants α, β, and γ so that this approximation is consistent with the Laplacian to fourth order? If so, derive the values that do so. If not, show why not.
b) Same question for the 9-point stencil on the right.

3. Write a python program to solve the 1-dimensional boundary value problem

$$-au''(x) + bu'(x) + cu(x) = f(x), \quad 0 \le x \le 1, \quad u(0) = g_0,\ u(1) = g_1.$$

Here the coefficients a, b, and c are given constants, as are the boundary values g_0 and g_1. Typically you should take $a > 0$ and $c \ge 0$, since this ensures there is a unique solution. Use a 3-point difference operator which is consistent to second order and make sure you store the matrix as a sparse matrix.

Choose a test problem for which your finite difference method is exact and check that your program gives the exact solution. Make sure that the problem tests all the terms (so none of a, b, or c should be zero). Then choose a problem for which the method is not exact put for which you know the exact solution, and compute the solution for a sequence of meshes, and find the rate of convergence. Finally find some more interesting problem and investigate the solution with your code.

For all problems hand in a complete, carefully written, well-justified description of your solution. For the third problem, email me a copy of your computer code in addition.