In this homework, you will make another application of scaling, here to prove what are called "inverse inequalities." Concerning notation, let $\mathcal{T}_h$ denote a triangulation of a plane domain Ω. For each triangle $T \in \mathcal{T}_h$, h_T denotes the diameter of T, ρ_T denotes the diameter of the inscribed disc in T, and $\sigma_T = h_T/\rho_T$ is the shape constant of T. We also denote by h the maximum of the h_T over all $T \in \mathcal{T}_h$, by σ_h the maximum of the σ_T, and by $h_{\min}$ the minimum of the h_T. We write $M_r(\mathcal{T}_h)$ for the space of all continuous piecewise polynomials of degree at most r with respect to the mesh $\mathcal{T}_h$ (the Lagrange finite element space of degree $r \geq 1$).

1. One inverse inequality has the form

$$\|u\|_{L^\infty(\Omega)} \leq C h_{\min}^{-1} \|u\|_{L^2(\Omega)}, \tag{1}$$

bounding the L^∞ norm of u in terms of its L^2 norm. There can be no such estimate for general smooth functions u, since it can easily happen that a function has an infinite L^∞ norm, but a bounded L^2 norms (like $u(x) = 1/|x|^{1/2}$ on any domain containing the origin). But you are to show that (1) holds for $u \in M^r(\mathcal{T}_h)$ with the constant C depending only on the shape regularity of the mesh (σ_h) and the degree r. State this precisely as a theorem, and prove it.

Hints: First prove the result in the case the mesh consists of only the reference triangle $\hat{T}$. This will be based on the equivalence of norms on a finite dimensional space. Next use scaling to handle the case of a mesh consisting of any single triangle. Here make clear how the dependence on the σ_T enters. Finally extend to a mesh of triangles.

2. In a similar way, precisely state and prove an inverse inequality that bounds the H^1 norm of a function $u \in M^r(\mathcal{T}_h)$ in terms of its L^2 norm.