1. Consider the following difference schemes for the advection equation $\partial u/\partial t + \partial u/\partial x = 0$ with periodic conditions on $[0, 2\pi]$ using a mesh size $h = 2\pi/N$ and a timestep $k > 0$.

- Lax–Friedrichs scheme: $\dfrac{U_n^{j+1} - (U_{n+1}^j + U_{n-1}^j)/2}{k} + \dfrac{U_{n+1}^j - U_{n-1}^j}{2h} = 0$
- backward/backward differences: $\dfrac{U_n^{j+1} - U_n^j}{k} + \dfrac{U_n^{j+1} - U_{n-1}^{j+1}}{h} = 0$
- backward/forward differences: $\dfrac{U_n^{j+1} - U_n^j}{k} + \dfrac{U_{n+1}^{j+1} - U_n^{j+1}}{h} = 0$
- backward/centered differences: $\dfrac{U_n^{j+1} - U_n^j}{k} + \dfrac{U_{n+1}^{j+1} - U_{n-1}^{j+1}}{2h} = 0$
- the box scheme:

$$\frac{(U_n^{j+1} + U_{n+1}^{j+1}) - (U_n^j + U_{n+1}^j)}{k} + \frac{(U_{n+1}^{j+1} + U_{n+1}^j) - (U_n^{j+1} + U_n^j)}{h} = 0$$

For each method:

(1) State whether the method is implicit or explicit.
(2) Compute the amplification matrix (i.e., the matrix G such that $U^{j+1} = GU^j$, where U^j is the vector of values at time jk). State it in terms of the shift matrices K_+ and K_-.
(3) Find the eigenvalues of the matrix G and determine under what conditions on h and k the method is stable. Flag each method as unconditionally stable, conditionally stable, or unconditionally unstable.

2. a) Show that box scheme approximates $(\partial u/\partial t + \partial u/\partial x)((n+1/2)h, (j+1/2)h)$ to within $O(k^2 + h^2)$.

b) [OPTIONAL] State precisely and prove a second order convergence result for the box scheme.