1. Besides the Raviart–Thomas family of mixed finite elements for the Poisson equation, there is a second family, called the BDM (Brezzi–Douglas–Marini) family. Let $r \geq 1$ denote the degree. For the vector variable in the mixed formulation, the BDM_r family uses the space V_h whose shape functions are the complete polynomial space $P_r(T; \mathbb{R}^2)$ and whose DOFs are:

- $\tau \mapsto \int_e \tau \cdot n_e \, p \, ds, \quad p \in \mathcal{P}_r(e), \quad e$ an edge,
  $\tau \mapsto \int_T \tau \cdot p^\perp \, dx, \quad p \in \mathcal{P}^-_{r-1}(T; \mathbb{R}^2)$ (if $r > 1$)

Draw the element diagrams for BDM_1 and BDM_2. Prove unisolvence for BDM_r (you may want to do $r = 1$ and 2 first to get the idea).

2. Define (for any triangulation $\mathcal{T}_h$) a projection operator $\pi_h : H^1(\Omega; \mathbb{R}^2) \to V_h$ (the BDM_r space) and prove the commuting diagram property

$$\operatorname{div} \pi_h \tau = P_h \operatorname{div} \tau, \quad \tau \in H^1(\Omega; \mathbb{R}^2).$$

Note: for the scalar variable in the mixed formulation, the BDM_r method uses the space W_h of all piecewise polynomials of degree $r - 1$, the same as for the Raviart–Thomas method, and here P_h denotes the L^2 projection into this space.

3. Prove that the BDM family of methods are stable.