Notes: It is great if you are able to use LaTeX. Then you can send me your solutions as a PDF file via email if you prefer. For the computer problem, send me the complete working python source code via email. For some of the problems you may find it helpful to use a symbolic algebra package like Mathematica or SymPy as an assistant for things like computing integrals and derivatives. That is fine, just explain how you obtained your results. You can discuss the homework with others, but everyone must write up their own solutions, and write their own computer code.

1. Consider the boundary value problem

$$(au')' = f \text{ in } I, \quad u(0) = u(1) = 0.$$

where $a : \bar{I} \to \mathbb{R}$ is a smooth everywhere positive coefficient function, and $f : \bar{I} \to \mathbb{R}$ a smooth function. We will use the finite difference approximation

$$(au')'(x) \approx L_h u(x) := \frac{1}{h}\left[a(x+h/2)\frac{u(x+h)-u(x)}{h} - a(x-h/2)\frac{u(x)-u(x-h)}{h}\right].$$

(a) Prove an $O(h^2)$ error estimate for the consistency error.
(b) Prove that the operator L_h acting on grid functions satisfies the discrete maximum principle.
(c) Write down the finite difference method and prove that it has a unique solution.

2. Consider the following possible stencils for finite difference approximations of the Laplacian.

$$\frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} & & \alpha & & \\ & & \beta & & \\ \alpha & \beta & \gamma & \beta & \alpha \\ & & \beta & & \\ & & \alpha & & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} \alpha & \beta & \alpha \\ \beta & \gamma & \beta \\ \alpha & \beta & \alpha \end{pmatrix}$$

The first one is the usual 5-point Laplacian and we have proved a second order bound on its consistency error:

$$\|\Delta_h u - \Delta u\|_{L^\infty(\Omega_h)} \le \frac{h^2}{12}(\|\partial^4 u/\partial x^4\|_{L^\infty(\bar{\Omega})} + \|\partial^4 u/\partial y^4/\|_{L^\infty(\bar{\Omega})}).$$

(a) Is it possible to choose the coefficients α, β, γ so that the finite difference method for the middle stencil satisfies a *fourth* order bound on its consistency error? If so, give the values of the coefficients and demonstrate that the consistency error is $O(h^4)$. If not, demonstrate that it is not possible.
(b) Same question for the right stencil.

3. Consider solving the problem $\Delta u = f$ by a finite difference method using a finite difference operator on both the left and the right of the equation: $L_h u_h = M_h f_h$. Taking as stencils for L_h and M_h

$$\frac{1}{6h^2}\begin{pmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{12}\begin{pmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{pmatrix}$$

respectively, show that the consistency error is $O(h^4)$ for a smooth solution u.

4. For the problem $-\Delta u = 1$ in Ω, $u = 0$ on $\partial\Omega$, with Ω the unit square:

(a) Compute the value of u at the center of the square *analytically* to at least six correct digits after the decimal point , using separation of variables (Fourier series). Supply your value and explain how you computed it and why you are confident that it is correct to six digits. (Hint: You may want to do the 1D problem first as a warm up.)

(b) Write a python computer code that uses the finite difference method to solve the boundary value problem with a uniform grid with mesh spacing $h = 1/4, 1/8, \ldots, 1/128$. Provide a table that shows for each mesh the approximation of $u(1/2, 1/2)$, the error in this quantity, and, for meshes after the first, the apparent rate of convergence.