1. Consider three different ways to express a classical iterative method:

$$u_{i+1} = u_i + B(f - Au_i), \quad u_{i+1} = P^{-1}(f - Qu_i), \quad u_{i+1} = Gu_i + Bf,$$

in terms of the approximate inverse B, the splitting matrix P, or the iteration matrix G. Make a table with four columns, one for the method and one each for B, P, and G. In the first column put the following five methods: *Richardson, Jacobi, Gauss–Seidel, damped Jacobi, SOR*, and then, in the remaining columns put in the formulas for the corresponding matrices. For example, in the row for Jacobi and the column for the iteration matrix G you should put $-D^{-1}(U + L)$, using the notations D, L, and U for the diagonal, strictly lower triangular, and strictly upper triangular parts of A.

2. Show that the following matrix is SPD, but that the Jacobi iteration does not converge for it (unlike Gauss–Seidel, which converges for all SPD matrices).

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

3. Consider the solution of $Ax = b$ for an SPD matrix A using the damped Jacobi iteration with damping parameter $\alpha \in [0, 1]$. Let G_α denote the corresponding iteration matrix.

(a) Give the formula for G_α in terms of α, D and L (where $A = L + D + L^T$). Write out in particular the formula for G_1 and check that it is the iteration matrix for the (undamped) Jacobi method.
(b) Prove that G_α has only real eigenvalues.
(c) Suppose that the minimum and maximum eigenvalues of G_1 (the iteration matrix for undamped Jacobi) are -2 and 0.5, respectively. What are the minimum and maximum values of the eigenvalues of G_α?
(d) For which values of $\alpha \in [0, 1]$ does the the damped Jacobi iteration converge?
(e) Which value of α optimizes the rate of convergence in this case, and what is the optimal rate?