One way to derive a mixed formulation is to minimize a quadratic energy function (the dual or complementary energy) subject to a linear constraint. The ingredients needed to carry this out are

- a pair of Hilbert spaces $V$ and $W$
- a quadratic functional $J : V \to \mathbb{R}$
- a bounded linear operator $L : V \to W^*$
- an element $f \in W^*$.

The problem is then to find a minimizer $\sigma$ of $J$ over $V$ subject to the constraint $L\sigma = f$. That is, we seek

$$\sigma \in V \text{ such that } L\sigma = f \text{ and } J(\sigma) = \min_{\substack{\tau \in V \\ L\tau = f}} J(\tau).$$

To solve the constrained minimization problem, we introduce a Lagrange multiplier $u \in W$ which converts the problem to that of finding a stationary point of a different functional, without any constraint. Specifically, we seek a pair $(\sigma, u) \in V \times W$ which is a stationary point of the quadratic functional

$$L : V \times W \to \mathbb{R}, \quad L(\tau, v) = J(\tau) + \langle L\tau - f, u \rangle.$$

We may then compute the corresponding Euler–Lagrange equations to get a system of linear equations that determine $\sigma$ and $u$.

**Example:** For example, let $\Omega$ be a domain in $\mathbb{R}^2$ and take

$$V = H(\operatorname{div}, \Omega), \quad W = L^2(\Omega), \quad L\tau = \operatorname{div} \tau, \quad f \in L^2(\Omega).$$

Note that, in this case, $W^* = L^2$, the same as $W$. Then the constrained minimization problem is to find $\sigma \in H(\operatorname{div})$ such that $\operatorname{div} \sigma = f$ and

$$\frac{1}{2}\|\sigma\|_{L^2}^2 = \min_{\substack{\tau \in H(\operatorname{div}) \\ \operatorname{div} \tau = f}} \frac{1}{2}\|\tau\|_{L^2}^2.$$

The Lagrange multiplier $u$ belongs to $L^2$ and the pair $(\sigma, u) \in H(\operatorname{div}) \times L^2$ is a stationary point of

$$L(\tau, v) = \frac{1}{2}\|\tau\|_{L^2}^2 + (\operatorname{div} \tau - f, v).$$

The Euler–Lagrange equations are then

$$(\sigma, \tau) + (\operatorname{div} \tau, u) = 0, \quad \tau \in H(\operatorname{div}), \qquad (\operatorname{div} \sigma, v) = (f, v), \quad v \in L^2.$$

This is a weak formulation of the boundary value problem

$$\sigma - \operatorname{grad} u = 0, \quad \operatorname{div} \sigma = f, \quad u = 0 \text{ on } \partial\Omega.$$

In this case we can eliminate $\sigma$ from the system and write a boundary value problem for the Lagrange multiplier variable $u$ alone: it satisfies the boundary value problem

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

which is the Dirichlet problem for the Poisson equation.

The homework assignment is to work out a few more examples in a way similar to the example above. Present each example carefully and clearly, defining the spaces and quadratic functionals involved, writing out the Euler–Lagrange equations in weak and strong form, etc.

1. Let $H(\operatorname{div}\operatorname{div})$ be the space of $2 \times 2$ symmetric-matrix-valued functions $M = M_{ij}$ for which

$$\operatorname{div}\operatorname{div} M := \sum_{i,j=1}^{2} \frac{\partial^2 M_{ij}}{\partial x^i \partial x^j}$$

belongs to $L^2$. Discuss (as above) the minimization of

$$J(M) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} |M_{ij}|^2\,dx \text{ subject to } \operatorname{div}\operatorname{div} M = f.$$

Be sure to state (among the other things) the boundary value problem satisfied by the Lagrange multiplier variable alone.

2. Consider minimizing the energy functional on $H^1$

$$J(v) = \frac{1}{2}\int_\Omega |\operatorname{grad} v|^2\,dx - \int_\Omega f v\,dx$$

subject to the constraint $\operatorname{tr} v = 0$. Here $\operatorname{tr} v = v|_{\partial\Omega}$ is the usual trace operator which maps $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ so the space $W$ should be taken to be $H^{-1/2}(\partial\Omega)$, the dual space of $H^{1/2}(\partial\Omega)$. This is an approach to solve the Dirichlet problem without building the boundary condition into the space (instead enforcing it through a Lagrange multiplier). In this case, it is not natural to reduce the problem to the Lagrange multiplier alone, but be sure to say how the Lagrange multiplier variable is related to the solution of the Dirichlet problem.

3. The non-mixed form of the biharmonic problem minimizes

$$\frac{1}{2}\int_\Omega |\operatorname{grad}\operatorname{grad} w|^2\,dx - \int_\Omega f w\,dx$$

over $w \in \mathring{H}^2$. One way to get a mixed method is to minimize

$$J(\tau, w) = \frac{1}{2}\int_\Omega |\operatorname{grad}\tau|^2\,dx - \int_\Omega f w\,dx,$$

over $V = \mathring{H}^1(\Omega;\mathbb{R}^2) \times \mathring{H}^1(\Omega)$ subject to the constraint $\tau - \operatorname{grad} w = 0$. Analyze this method in the framework above.