Math 8446 study problems

1. Consider the solution of the heat equation on the unit square by finite differences, using forward differences for time discretization. What condition on the mesh spacing h and time step k is needed for stability in the maximum norm? Explain.

2. State the Crank–Nicolson scheme for the 1D heat equation using grid spacing h and time step k. Write out the matrix for the linear system that is to be solved at each time step. What are its eigenvalues? What conditions on h and k do we need to insure convergence when $h \to 0$ and $k \to 0$? What is the principal advantage to Crank–Nicolson as compared to the backward Euler method?

3. Give an example of a consistent finite difference method for the advection equation $\partial u/\partial t+\partial u/\partial x$ which satisfies the CFL condition for some values of h and k but not others and state the requirement that the CFL condition imposes on h and k.

4. Is the CFL condition sufficient for stability of a consistent finite difference scheme for the advection equation? Justify or give a counterexample.

5. Explain how finite element discretization in space of the heat equation leads to a system of ordinary differential equations in time.

6. Let $u(x, t)$ solve the heat equation for x in a polygonal domain Ω and $t \in [0, T]$, subject to Neumann boundary conditions and an initial condition. Write down the finite element semidiscretization in space and state and prove a second order convergence estimate.

7. For each of the following triangular finite elements give the shape functions and the degrees of freedom and prove unisolvence. For each determine whether the assembled finite element space is contained in H^1 and whether it is contained in H^2.

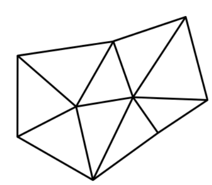
a) The Hermite cubic finite element

b) The reduced Hermite quintic finite element

c) The Morley finite element

8.

What is the dimension of the space of Hsieh-Clough-Tocher finite elements satisfying the boundary conditions $u = \partial u/\partial n = 0$ on the following mesh?



9. Given a triangulation $\mathcal{T}_h$ of a domain Ω, let V_h denote the nonconforming P_1 (Crouzeix–Raviart) finite element approximation of $H^1(\Omega)$. Suppose that u solves $-\Delta u + cu = f$ in Ω with Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$. Define

$$E_h(u, v) = \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{grad} u \cdot \operatorname{grad} v + cuv)\, dx - \int_\Omega f v\, dx.$$

Derive a formula for $E_h(u, v)$ as a sum of integrals over edges in the mesh. Define all the terms entering the formula precisely.

10. a) For the spaces V_h of Lagrange piecewise linear finite elements and W_h of Crouzeix–Raviart nonconforming piecewise linear finite elements (on the same triangulation in 2D), does one space contain the other, and if so which? Which space has the larger dimension?

b) Answer the same questions with V_h the space of Lagrange piecewise quadratic finite elements and W_h the Morley space.

11. Consider defining a finite element on a triangle with shape function space $\mathcal{P}_2$ and degrees of freedom the values of the function at two points on each edge, namely the 1/3 and 2/3 of the way across the edge. Prove that these degrees of freedom are *not* unisolvent.

12. Give the mixed variational formulation for the PDE problem

$$-\operatorname{div} a \operatorname{grad} u = f \text{ in } \Omega,$$

for each of the boundary conditions $\quad u = g \quad$ or $\quad \partial u/\partial n = 0$. (For the latter assume $\int f = 0$ and normalize the solution by $\int u = 0$.)

13. Suppose we discretize the Dirichlet problem for the Laplacian in one dimension ($-u'' = f$ on $(0,1)$, $u(0) = u(1) = 0$), using the mixed formulation and Lagrange quadratic finite elements both for u and $\sigma = u'$. Prove that the resulting matrix is singular.

14. Given a triangulation of a domain Ω in the plane, let V_h be the lowest order Raviart–Thomas space and W_h the space of piecewise constants.

a) Define the operator $\pi_h : H^1(\Omega; \mathbb{R}^2) \to V_h$ and prove that it satisfies

$$\int_\Omega \operatorname{div} \pi_h \tau \, v \, dx = \int_\Omega \operatorname{div} \tau \, v \, dx, \quad v \in W_h.$$

b) State the inf-sup condition (second Brezzi condition) required for stability of these elements for the mixed Poisson problem.

c) Use the π_h operator to prove the inf-sup condition (you can assume that π_h is bounded uniformly in h.

15. Consider the mixed method for the 1D Laplacian on an interval Ω.

a) Write down the bilinear forms $a : H^1 \times H^1 \to \mathbb{R}$ and $b : H^1 \times L^2 \to \mathbb{R}$ and in terms of them state the two Brezzi stability conditions for subspaces $V_h \subset H^1$ and $W_h \subset L^2$.

b) Suppose that for the subspace V_h we choose the Lagrange piecewise quadratics and for the subspace W_h we choose piecewise constants. One of the two stability conditions holds in this case and one fails. Which one holds and which one fails? Prove your claim.