Math 8446 study problems

1. Give the mixed variational formulation for the PDE problem

$$-\operatorname{div} a \operatorname{grad} u = f \text{ in } \Omega,$$

for each of the boundary conditions $u = g$ or $\partial u/\partial n = 0$. (For the latter assume $\int f = 0$ and normalize the solution by $\int u = 0$.)

2. Suppose we discretize the Dirichlet problem for the Laplacian in one dimension ($-u'' = f$ on $(0,1)$, $u(0) = u(1) = 0$), using the mixed formulation and Lagrange quadratic finite elements both for u and $\sigma = u'$. Prove that the resulting matrix is singular.

3. Given a triangulation of a domain Ω in the plane, let V_h be the lowest order Raviart–Thomas space and W_h the space of piecewise constants.

a) Define the operator $\pi_h : H^1(\Omega; \mathbb{R}^2) \to V_h$ and prove that it satisfies

$$\int_\Omega \operatorname{div} \pi_h \tau \, v \, dx = \int_\Omega \operatorname{div} \tau \, v \, dx, \quad v \in W_h.$$

b) State the inf-sup condition (second Brezzi condition) required for stability of these elements for the mixed Poisson problem.

c) Use the π_h operator to prove the inf-sup condition (you can assume that π_h is bounded uniformly in h.

4. Consider the mixed method for the 1D Laplacian on an interval Ω.

a) Write down the bilinear forms $a : H^1 \times H^1 \to \mathbb{R}$ and $b : H^1 \times L^2 \to \mathbb{R}$ and in terms of them state the two Brezzi stability conditions for subspaces $V_h \subset H^1$ and $W_h \subset L^2$.

b) Suppose that for the subspace V_h we choose the Lagrange piecewise quadratics and for the subspace W_h we choose piecewise constants. One of the two stability conditions holds in this case and one fails. Which one holds and which one fails? Prove your claim.

5. The lowest order Brezzi–Douglas–Marini finite element subspace V_h of $H(\operatorname{div})$ uses as shape functions $\mathcal{P}_1(T; \mathbb{R}^2)$, T being a triangle.

a) What are the degrees of freedom associated to a vertex, to an edge, and to the triangle? Draw the element diagram.

b) Prove unisolvence.

c) Define a projection operator $\pi_h : H^1(\Omega; \mathbb{R}^2) \to V_h$ satisfying a commutativity property involving the operator div. State and prove the commutativity property.

6. a) For the mini finite element method for the Stokes equations construct a Fortin operator. State the two properties such an operator must have and sketch the proof of these properties.

b) What is the basic error estimate for solution of the Stokes equations using these elements (proof not required)?

7. Same question for the $\mathcal{P}_2 - \mathcal{P}_0$ Stokes elements.

8. Ω be the unit cube in $\mathbb{R}^3$ and consider the closed Hilbert complex

$$L^2(\Omega;\mathbb{R}^3) \xrightarrow{(\mathrm{curl}, H(\mathrm{curl}))} L^2(\Omega;\mathbb{R}^3) \xrightarrow{(\mathrm{div}, H(\mathrm{div}))} L^2(\Omega).$$

a) Write down the mixed weak formulation of the Hodge Laplacian corresponding to this situation.

b) Write down the corresponding strong formulation including boundary conditions.

9. Let $W^0 = L^2(\Omega;\mathbb{R}^2)$ and $W^1 = L^2(\Omega)$. On W^1 use the usual L^2 norm, but on W^0 use the weighted norm

$$\|\tau\|^2_{W^0} = \int_\Omega a(x)|\tau(x)|^2\,dx$$

where $a : \Omega \to \mathbb{R}$ is a given continuous coefficient function with $0 < \underline{a} \le a(x) \le \bar{a}(x)$. Let T be the operator $W^0 \to W^1$ given by $T\tau = \mathrm{div}\,\tau$ with $D(T) = H(\mathrm{div})$, and consider the Hilbert complex

$$W^0 \xrightarrow{T} W^1 \to 0.$$

a) Explain briefly why W^1 contains no harmonic forms in this case.

b) Write out the mixed weak formulation of the Hodge Laplacian for this complex.

c) What is the corresponding boundary value problem (PDE and boundary conditions).

10. For a closed Hilbert complex

$$\cdots \to W^{k-1} \xrightarrow{(d^{k-1}, V^{k-1})} W^k \xrightarrow{(d^k, V^k)} W^{k+1} \to \cdots,$$

a) Define the harmonic forms $\mathfrak{H}^k$.

b) State and prove the Hodge decomposition of W^k.

c) Define the abstract Hodge Laplacian L^k.

d) For a function $f \in W^k$ which is orthogonal to the harmonic form, express the Hodge decomposition of f in terms of a solution u to the equation $L^k u = f$.

11. Let Ω be a simply-connected domain in $\mathbb{R}^2$ and consider the Hilbert complex

$$L^2(\Omega) \xrightarrow{(\mathrm{curl}, \mathring{H}^1)} L^2(\Omega;\mathbb{R}^2) \xrightarrow{(\mathrm{div}, \mathring{H}(\mathrm{div}))} L^2(\Omega).$$

(Here $\mathrm{curl}\,\phi = (\partial\phi/\partial y, -\partial\phi/\partial x)$.)

a) Write down the mixed weak formulation of Hodge Laplacian.

b) State the PDE and boundary conditions in terms of the primary variable $u : \Omega \to \mathbb{R}^2$.

12. Recall that an element $\omega \in \mathrm{Alt}^1\,\mathbb{R}^n$ has a vector proxy $w \in \mathbb{R}^n$ given by $\omega(v) = w \cdot v$ for all $v \in \mathbb{R}^n$. (w is the vector proxy of ω). Now suppose that $L : \mathbb{R}^m \to \mathbb{R}^n$. Define the pullback $L^*\omega \in \mathrm{Alt}^1\,\mathbb{R}^m$ and prove that the vector proxy of $L^*\omega$ is $L^T w \in \mathbb{R}^m$.