(1) The Gauss-Seidel method converges for every SPD matrix, but the same is not true for the Jacobi method. In this problem we will investigate an example. Let $A$ be the matrix given by $a_{ii} = 1 + c$, $a_{ij} = c$ for $i \neq j$, where $c$ is a positive constant. First, find all the eigenvalues of $A$ and conclude that it is SPD for all $c \geq 0$ (you may wish to first consider $A - I$). Second determine for which values of $c$ the Jacobi method for $Au = f$ converges. Justify your results.

(2) Let $P_n$ denote the space of polynomials in one variable of degree at most $n$. The analysis of the conjugate gradient method depends on the following fact. For any $0 < a < b$ and integer $n > 0$,

$$
\min_{p \in P_n} \max_{x \in [a,b]} |p(x)| = \frac{2}{\left(\frac{\sqrt{b/a+1}}{\sqrt{b/a-1}}\right)^n + \left(\frac{\sqrt{b/a-1}}{\sqrt{b/a+1}}\right)^n}.
$$

Write a clear and complete proof of this fact, based on the following ideas. The Chebyshev polynomials, are defined by the recursion

$$
T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for } n = 1, 2, \ldots.
$$

Two explicit formulas for $T_n(x)$ are

$$
T_n(x) = \cos(n \arccos x), \quad T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n],
$$

with the first equation valid for $|x| \leq 1$ and the second valid for $|x| \geq 1$. $T_n$ is a polynomial of degree $n$ and satisfies $|T_n(x)| \leq 1$ on $[-1,1]$ with equality holding for $n + 1$ distinct numbers in $[-1,1]$. $T_n$ minimizes of max$_{x \in [-1,1]} |p(x)|$ over all polynomials in $P_n$ which take the value $T_n(\alpha)$ at $\alpha$. The polynomial

$$
p(x) = \left[T_n \left(-\frac{b+a}{b-a}\right)\right]^{-1} T_n \left(\frac{2x - b - a}{b-a}\right)
$$

solves the minimization problem above.

(3) This is a computational problem to be carried out with Matlab (if you want to use different software, discuss with me). Consider the boundary value problem

$$
-\epsilon u'' + u' = 1 \text{ on } (0,1), \quad u(0) = u(1) = 0.
$$

where $\epsilon > 0$. Verify that the exact solution is

$$
u(x) = x - \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}.
$$

Plot the solution for $\epsilon = 0.1, 0.01, \text{ and } 0.001$ (you may need to be clever to avoid overflow for $\epsilon = 0.001$).

Compute and plot the solution using the 3-point difference method (with centered differences for the $u'$ term), and a uniform mesh of size $h = 1/n$. Investigate the
convergence of the method as $h$ decreases using the maximum norm and the norm

$$\|e\|_p = \left( h \sum_{i=1}^{n} |e(ih)|^p \right)^{1/p},$$

for $p = 1$ and 2. Describe the convergence behavior and how it varies for the three values of $\epsilon$.

Next, let $m = n/2$ (assume $n$ is even) and put $h_2 = \bar{x}/m$, $h_2 = (1 - \bar{x})/m$, where $\bar{x} \in (0, 1)$ is to be defined. Now we define mesh points by

$$x_i = ih_1, \quad i = 0, 1, \ldots, m, \quad x_{m+i} = \bar{x} + ih_2, \quad i = 1, 2, \ldots, m.$$

Thus we use mesh spacing $h_1$ for the points $0 = x_0 < x_1 < \ldots < x_m = \bar{x}$ and mesh spacing $h_2$ for $\bar{x} = x_m < x_{m+1} < \ldots < x_n = 1$. For the problem above we take $\bar{x} = 1 - \epsilon |\log \epsilon|$. The resulting mesh is called a *Shishkin mesh*. Implement the 3-point difference method to solve our problem on this mesh and again study the convergence. Be sure to implement the 3-point difference operator correctly at the mesh point $x_m = \bar{x}$, which has unequally spaced neighbors.

Report all your results clearly remembering Hamming’s dictum that “the purpose of computing is insight, not numbers.” Use graphics and tables if they help to clarify the results. Include a printout of your source code.