Finite Element Exterior Calculus and Applications

Part I

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Primary references

Two (long) papers with R. Falk and R. Winther:

Finite element exterior calculus, homological techniques, and applications.
Acta Numer 15 (2006) pp. 1-155.

Finite element exterior calculus: from Hodge theory to numerical stability.
Bull. AMS 47 (2010) pp. 281-354.

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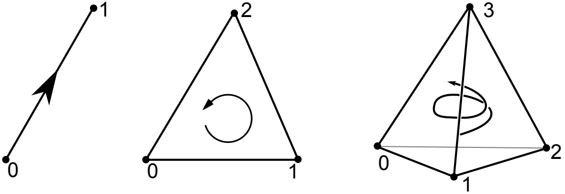
Basic homology

Chain complexes

- *Chain complex:* seq. of vector spaces and linear maps
  $\cdots \to V_{k+1} \xrightarrow{\partial_{k+1}} V_k \xrightarrow{\partial_k} V_{k-1} \to \cdots$ with $\partial_k \circ \partial_{k+1} = 0$.
- Alternative viewpoint: $V = \bigoplus_k V_k$ is a *graded vector space* and $\partial : V \to V$ is a *graded linear operator* of degree -1 such that $\partial \circ \partial = 0$
- V_k: k-chains
  $\mathfrak{Z}_k = \mathcal{N}(\partial_k)$: k-cycles
  $\mathfrak{B}_k = \mathcal{R}(\partial_{k+1})$: k-boundaries
  $\mathcal{H}_k = \mathfrak{Z}_k / \mathfrak{B}_k$: k-th homology space

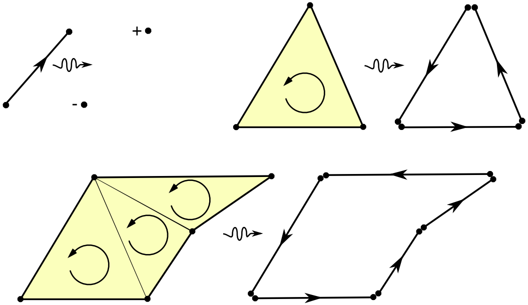
Simplicial complexes

- A *k-simplex* in $\mathbb{R}^n$ is the convex hull $f = [x_0, \ldots, x_k]$ of $k+1$ vertices in general position.
- A subset determines a *face* of f: $[x_{i_0}, \ldots, x_{i_d}]$.
- *Simplicial complex:* A finite set $\mathcal{S}$ of simplices in $\mathbb{R}^n$, such that
  1. Faces of simplices in $\mathcal{S}$ are in $\mathcal{S}$.
  2. If $f \cap g \neq \emptyset$ for $f, g \in \mathcal{S}$, then it is a face of f and of g.
- If we order all vertices of $\mathcal{S}$, then an ordering of the vertices of the simplex determines an *orientation*.



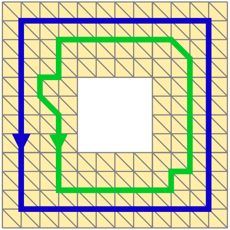
The boundary operator on chains

$\Delta_k(\mathcal{S})$:	the set of k-simplices in $\mathcal{S}$
C_k (k-chains):	formal linear combinations $c = \sum_{f \in \Delta_k(\mathcal{S})} c_f f$
$\partial_k : \Delta_k \to C_{k-1}$:	$\partial[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} (-1)^i [\ldots, \hat{x}_i, \ldots]$
$\partial_k : C_k \to C_{k-1}$:	$\partial c = \sum c_f \, \partial f$



The simplicial chain complex

Every simplicial complex gives rise to an associated chain complex.



$$0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0$$

$\beta_k := \dim \mathcal{H}_k(C)$ is the kth Betti number

1, 1, 0, 0

1, 1, 0, 0

1, 2, 1, 0

2, 5 , 0, 0

1, 0, 1, 0

Chain maps

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V_{k+1} & \xrightarrow{\partial_{k+1}} & V_k & \xrightarrow{\partial_k} & V_{k-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\
\cdots & \longrightarrow & V'_{k+1} & \xrightarrow{\partial'_{k+1}} & V'_k & \xrightarrow{\partial'_k} & V'_{k-1} & \longrightarrow & \cdots
\end{array}$$

- $f(\mathfrak{Z}) \subset \mathfrak{Z}', f(\mathfrak{B}) \subset \mathfrak{B}'$, so f induces $\bar{f} : \mathcal{H}(V) \to \mathcal{H}(V')$.
- If V' is a subcomplex ($V'_k \subset V_k$ and $\partial' = \partial|_V$), and $fv = v$ for $v \in V'$, we call f a *chain projection*.

PROPOSITION

A chain projection induces a surjection on homology.

Cochain complexes

A cochain complex is like a chain complex but with *increasing* indices.

$$\cdots \to V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \to \cdots$$

- cocycles $\mathfrak{Z}^k$, coboundaries $\mathfrak{B}^k$, cohomology $\mathcal{H}^k$, ...
- The dual of a chain complex is a cochain complex:

$$\partial_{k+1} : V_{k+1} \to V_k \quad \Longrightarrow \quad \partial^*_{k+1} : V^*_k \to V^*_{k+1}$$

d^k V^k

The de Rham complex for a domain in $\mathbb{R}^n$

1-D: $$0 \to C^\infty(\Omega) \xrightarrow{d/dx} C^\infty(\Omega) \to 0$$

2-D: $$0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(\Omega) \to 0$$

3-D: $$0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$$

n-D: $$0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$$

The space $\Lambda^k(\Omega) = C^\infty(\Omega, \mathbb{R}^{n\times\cdots\times n}_{\text{skw}})$, the space of smooth *differential k-forms* on Ω.

- *Exterior derivative:* $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$
- *Integral of a k-form over an oriented k-simplex:* $\int_f v \in \mathbb{R}$
- *Stokes theorem:* $\int_c du = \int_{\partial c} u, \quad u \in \Lambda^{k-1}, c \in C_k$
- All this works on *any smooth manifold*

De Rham's Theorem

- De Rham map:

$$\begin{array}{ccc} \Lambda^k(\Omega) & \longrightarrow & C^k(\mathcal{S}) := C_k(\mathcal{S})^* \\ u & \longmapsto & \left(c \mapsto \int_c u\right) \end{array}$$

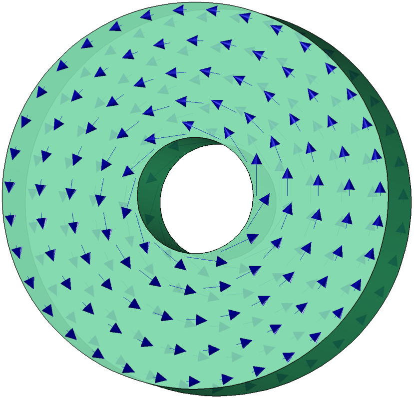
- Stokes theorem says it's a cochain map, so induces a map from de Rham to simplicial cohomology.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & \Lambda^k(\Omega) & \xrightarrow{d} & \Lambda^{k+1}(\Omega) & \xrightarrow{d} \\ & & \downarrow & & \downarrow & \\ \cdots & \xrightarrow{\partial^*} & C^k & \xrightarrow{\partial^*} & C^{k+1} & \xrightarrow{\partial^*} \end{array}$$

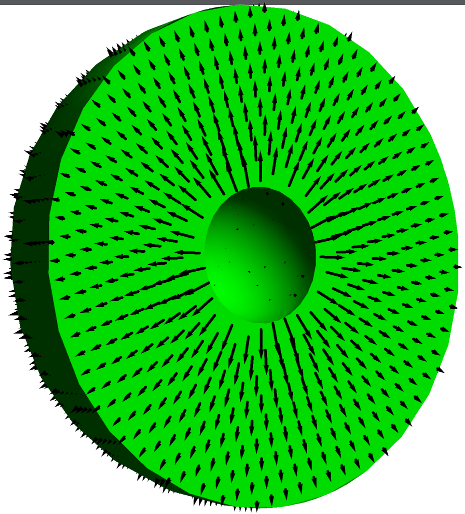
Theorem (De Rham's theorem)

The induced map is an isomorphism on cohomology.

Nonzero cohomology classes



$u = \operatorname{grad}\theta, 0 \neq \bar{u} \in \mathcal{H}^1$
on cylindrical shell



$u = \operatorname{grad}\dfrac{1}{r}, 0 \neq \bar{u} \in \mathcal{H}^2$
on spherical shell

Unbounded operators on Hilbert space

Unbounded operators

- X,Y H-spaces (extensions to Banach spaces, TVSs,...)
- $T : D(T) \to Y$ linear, $D(T) \subseteq X$ subspace (not necessarily closed), T not necessarily bounded
- *Not-necessarily-everywhere-defined-and-not-necessarily-bounded linear operators*
- *Densely defined:* $\overline{D(T)} = X$
- Ex: $X = L^2(\Omega)$, $Y = L^2(\Omega;\mathbb{R}^n)$, $D(T) = H^1(\Omega)$, $Tv = \operatorname{grad} v$ (changing $D(T)$ to $\mathring{H}^1(\Omega)$ gives a *different* example)
- S, T unbdd ops $X \to Y \implies D(S+T) = D(S) \cap D(T)$ (may not be d.d.)
- $X \xrightarrow{S} Y, Y \xrightarrow{T} Z$ unbdd ops $\implies D(T \circ S) = \{v \in D(S) \mid Sv \in D(T)\}$
- *Graph norm (and inner product):* $\|v\|^2_{D(T)} := \|v\|^2_X + \|Tv\|^2_Y, v \in D(T)$
- *Null space, range, graph:* $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\Gamma(T)$

Closed operators

- T is *closed* if $\Gamma(T)$ is closed in $X \times Y$.
- Equivalent definitions:
  1. If $v_1, v_2, \ldots \in D(T)$ satisfy $v_n \xrightarrow{X} x$ and $Tv_n \xrightarrow{Y} y$ for some $x \in X$ and $y \in Y$, then $x \in D(T)$ and $Tx = y$.
  2. $D(T)$ endowed with the graph norm is complete.
- If $D(T) = X$, then T is closed $\iff$ T is bdd (Closed Graph Thm)

Many properties of bounded operators extend to closed operators. E.g.,

PROPOSITION

Let T be a closed operator X to Y.

1. $\mathcal{N}(T)$ *is closed in* X.
2. $\exists \gamma > 0$ *s.t.* $\|Tx\|_Y \geq \gamma \|x\|_X \iff \mathcal{N}(T) = 0, \mathcal{R}(T)$ *closed*
3. *If* $\dim Y / \mathcal{R}(T) < \infty$, *then* $\mathcal{R}(T)$ *is closed*

Adjoint of a d.d.unbdd operator

Let T be a d.d.unbdd operator $X \to Y$. Define

$D(T^*) = \{w \in Y \mid \text{the map } v \in D(T) \mapsto \langle w, Tv \rangle_Y \in \mathbb{R} \text{ is bdd in } X\text{-norm}\}$

For $w \in D(T^*)$ $\quad \exists!\, T^*w \in X$ s.t.

$$\langle T^*w, v \rangle_X = \langle w, Tv \rangle_Y, \quad v \in D(T),\ w \in D(T^*).$$

T^* is a closed operator (even if T is not). Define the rotated graph

$$\tilde{\Gamma}(T^*) = \{\, (-T^*w, w) \mid w \in D(T^*) \,\} \subset X \times Y,$$

Then $\Gamma(T)^\perp = \tilde{\Gamma}(T^*)$, $\overline{\Gamma(T)} = \tilde{\Gamma}(T^*)^\perp$.

Proposition

Let T be a closed d.d. operator $X \to Y$. Then

1. *T^* is closed d.d.*
2. *$T^{**} = T$.*
3. $$\mathcal{R}(T)^\perp = \mathcal{N}(T^*), \quad \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)},$$
   $$\mathcal{R}(T^*)^\perp = \mathcal{N}(T), \quad \mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}.$$

Closed Range Theorem

THEOREM

Let T be a closed d.d.operator $X \to Y$. If $\mathcal{R}(T)$ is closed in Y, then $\mathcal{R}(T^*)$ is closed in X.

Proof.

1. Reduce to case T is surjective.
2. Restrict to orthog comp of $\mathcal{N}(T)$ in $D(T)$ (w/ graph norm). Get bounded linear isomorphism. $\exists$ bounded inverse:

$$\forall y \in Y \ \exists x \in X \text{ s.t. } Tx = y, \quad \|x\|_X \leq c\|y\|_Y$$

3. This implies $\|y\|_Y \leq c\|T^*y\|_X$, $y \in D(T^*)$. □

Grad, curl, and div

Assume $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary (so trace theorem holds).

- (grad, H^1) is closed. Its adjoint is $(-\text{div}, \mathring{H}^1)$.
- $(\text{curl}, H(\text{curl}))$ is closed, with adjoint $(\text{curl}, \mathring{H}(\text{curl}))$
- $(\text{div}, H(\text{div}))$ is closed, with adjoint $(-\text{grad}, \mathring{H}^1)$

Hilbert complexes

Hilbert complexes

Definition

A *Hilbert complex* is a sequence of Hilbert spaces W^k and a sequence of closed d.d.linear operators d^k from W^k to W^{k+1} such that $\mathcal{R}(d^k) \subset \mathcal{N}(d^{k+1})$.

- $V_k = D(d^k)$ H-space with graph norm: $\|v\|^2_{V^k} = \|v\|^2_{W^k} + \|d^k v\|^2_{W^{k+1}}$
- The *domain complex*

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^n \to 0$$

  is a *bounded* Hilbert complex (with less information).
- It is a cochain complex, so it has (co)cycles, boundaries, and homology.
- An H-complex is *closed* if $\mathfrak{B}^k$ is closed in W^k (or V^k).
- An H-complex is *Fredholm* if $\dim \mathcal{H}^k < \infty$.

$$\text{Fredholm} \implies \text{closed}$$

The dual complex

Define $d_k^* : V_k^* \subset W^k \to W^{k-1}$ as the adjoint of $d^{k-1} : V^k \subset W^{k-1} \to W^k$.

It is closed d.d.and, since $\mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,

$$\mathcal{R}(d_{k+1}^*) \subset \overline{\mathcal{R}(d_{k+1}^*)} = \mathcal{N}(d^k)^\perp \subset \mathcal{R}(d^{k-1})^\perp = \mathcal{N}(d_*^k),$$

so we get a Hilbert *chain* complex with domain complex

$$0 \to V_n^* \xrightarrow{d_n^*} V_{n-1}^* \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_1^*} V_0^* \to 0.$$

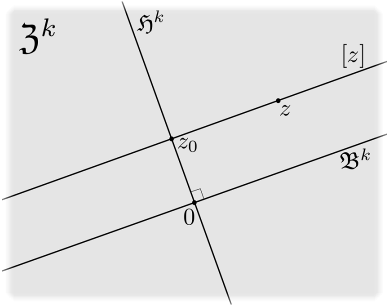
If (W, d) is closed, then (W, d^*) is as well, by the Closed Range Theorem.

From now on we mainly deal with closed H-complexes...

Harmonic forms

The Hilbert structure of a *closed* H-complex allows us to identify the homology space $\mathcal{H}^k = \mathfrak{Z}^k / \mathfrak{B}^k$ with a subspace $\mathfrak{H}^k$ of W^k:

$$\mathfrak{H}^k := \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} = \mathfrak{Z}^k \cap \mathfrak{Z}_k^* = \{u \in V^k \cap V_k^* \mid du = 0,\ d^*u = 0\}.$$



An *H*-complex has the *compactness property* if $V^k \cap V_k^*$ is dense and *compact* in W^k. This implies $\dim \mathfrak{H}^k < \infty$.

$$\text{compactness property} \implies \text{Fredholm} \implies \text{closed}$$

Two key properties of closed H-complexes

THEOREM (HODGE DECOMPOSITION)

For any closed Hilbert complex:

$$W^k = \underbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k}_{\mathfrak{Z}^k} \oplus \underbrace{\mathfrak{B}^*_k}_{\mathfrak{Z}^{k\perp}}$$

$$V^k = \overbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k} \oplus \mathfrak{Z}^{k\perp_V}$$

THEOREM (POINCARÉ INEQUALITY)

For any closed Hilbert complex, $\exists$ a constant c^P s.t.

$$\|z\|_V \leq c^P \|dz\|, \quad z \in \mathfrak{Z}^{k\perp_V}.$$

L^2 de Rham complex on $\Omega \subset \mathbb{R}^3$

k	W^k	d^k	V^k	d_k^*	V_k^*	$\dim \mathfrak{H}^k$
0	$L^2(\Omega)$	grad	H^1	0	L^2	β_0
1	$L^2(\Omega;\mathbb{R}^3)$	curl	$H(\mathrm{curl})$	$-\,\mathrm{div}$	$\mathring{H}(\mathrm{div})$	β_1
2	$L^2(\Omega;\mathbb{R}^3)$	div	$H(\mathrm{div})$	curl	$\mathring{H}(\mathrm{curl})$	β_2
3	$L^2(\Omega)$	0	L^2	$-\,\mathrm{grad}$	$\mathring{H}^1$	0

$$0 \to H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0$$

$$0 \leftarrow L^2 \xleftarrow{-div} \mathring{H}(\mathrm{div}) \xleftarrow{\mathrm{curl}} \mathring{H}(\mathrm{curl}) \xleftarrow{-\,\mathrm{grad}} \mathring{H}^1 \leftarrow 0$$

The abstract Hodge Laplacian

- $W^{k-1} \underset{d^*}{\overset{d}{\rightleftarrows}} W^k \underset{d^*}{\overset{d}{\rightleftarrows}} W^{k+1} \qquad L := d^*d + dd^* \qquad W^k \xrightarrow{L} W^k$
- $D(L^k) = \{\, u \in V^k \cap V_k^* \,|\, du \in V_{k+1}^*,\ d^*u \in V^{k-1} \,\}$
- $\mathcal{N}(L^k) = \mathfrak{H}^k, \quad \mathfrak{H}^k \perp \mathcal{R}(L^k)$
- *Strong formulation:* Find $u \in D(L^k)$ s.t. $Lu = f - P_{\mathfrak{H}} f, \quad u \perp \mathfrak{H}$.
- *Primal weak formulation:* Find $u \in V^k \cap V_k^* \cap \mathfrak{H}^{k\perp}$ s.t.

$$\langle du, dv \rangle + \langle d^*u, d^*v \rangle = \langle f - P_{\mathfrak{H}} f, v \rangle,\ v \in V^k \cap V_k^* \cap \mathfrak{H}^{k\perp}.$$

- *Mixed weak formulation.* Find $\sigma \in V^{k-1}$, $u \in V^k$, and $p \in \mathfrak{H}^k$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, && \tau \in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, && v \in V^k, \\ \langle u, q \rangle &= 0, && q \in \mathfrak{H}^k. \end{aligned}$$

Equivalence and well-posedness

THEOREM

Let $f \in W^k$. Then $u \in W^k$ solves the strong formulation $\iff$ it solves the primal weak formulation. Moreover, in this case, if we set $\sigma = d^*u$ and $p = P_{\mathfrak{H}}u$, then the triple (σ, u, p) solves the mixed weak formulation. Finally, if some (σ, u, p) solves the mixed weak formulation, then $\sigma = d^*u$, $p = \mathcal{P}_{\mathfrak{H}}u$, and u solves the strong and primal formulations of the problem.

THEOREM

For each $f \in W^k$ there exists a unique solution. Moreover

$$\|u\| + \|du\| + \|d^*u\| + \|dd^*u\| + \|d^*du\| \le c\|f - P_{\mathfrak{H}}f\|.$$

The constant depends only on the Poincaré inequality constant c^P.

Proof of well-posedness

We used the mixed formulation. Set

$$B(\sigma,u,p;\tau,v,q) = \langle \sigma,\tau\rangle - \langle u, d\tau\rangle - \langle d\sigma, v\rangle - \langle du, dv\rangle - \langle p, v\rangle - \langle u, q\rangle$$

We must prove the inf-sup condition: $\forall\ (\sigma,u,p)\ \exists\ (\tau,v,q)$ s.t.

$$B(\sigma,u,p;\tau,v,q) \geq \gamma(\|\sigma\|_V + \|u\|_V + \|p\|)(\|\tau\|_V + \|v\|_V + \|q\|),$$

with $\gamma = \gamma(c^P) > 0$. Via the Hodge decomposition,

$$u = u_{\mathfrak{B}} + u_{\mathfrak{H}} + u_{\mathfrak{B}^*} = d\rho + u_{\mathfrak{H}} + u_{\mathfrak{B}^*}$$

with $\rho \in \mathfrak{Z}^{\perp_V}$. Then take

$$\tau = \sigma - \frac{1}{(c^P)^2}\rho, \quad v = -u - d\sigma - p, \quad q = p - u_{\mathfrak{H}}.$$

Hodge Laplacian and Hodge decomposition

- $f = dd^*u + P_{\mathfrak{H}}f + d^*du$ is the Hodge decomposition of f
- Define $K : W^k \to D(L^k)$ by $Kf = u$ (bdd lin op).
- $P_{\mathfrak{B}} = dd^*K, \quad P_{\mathfrak{B}^*} = d^*dK$
- If $f \in V$, then $Kdf = dKf$.
- If $f \in \mathfrak{B}$, then $dKf = 0$. Since $Kf \perp \mathfrak{H}$, $Kf \in \mathfrak{B}$.
- $\mathfrak{B}$ *problem:* If $f \in \mathfrak{B}$, then $u = Kf$ solves

$$dd^*u = f, \quad du = 0, \quad u \perp \mathfrak{H}.$$

- $\mathfrak{B}^*$ *problem:* If $f \in \mathfrak{B}^*$, then $u = Kf$ solves

$$d^*du = f, \quad d^*u = 0, \quad u \perp \mathfrak{H}.$$

The Hodge Laplacian on a domain in 3D

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \to 0$$

$$0 \leftarrow L^2 \xleftarrow{-div} \mathring{H}(\text{div}) \xleftarrow{\text{curl}} \mathring{H}(\text{curl}) \xleftarrow{-\text{grad}} \mathring{H}^1 \leftarrow 0$$

k	$L^k = d^*d + dd^*$	BCs imposed on…		$V^{k-1} \times V^k$
0	$-\Delta$		$\partial u/\partial n$	H^1
1	$\text{curl}\,\text{curl} - \text{grad}\,\text{div}$	$u \cdot n$	$\text{curl}\, u \times n$	$H^1 \times H(\text{curl})$
2	$-\text{grad}\,\text{div} + \text{curl}\,\text{curl}$	$u \times n$	$\text{div}\, u$	$H(\text{curl}) \times H(div)$
3	$-\Delta$	u		$H(\text{div}) \times L^2$

essential BC for primal form. natural BC for primal form.

The Hodge wave equation

$$\ddot{U} + (dd^* + d^*d)U = 0, \quad U(0) = U_0, \quad \dot{U}(0) = U_1$$

Then $\sigma := d^*U,\ \rho := dU,\ u := \dot{U}$ satisfy

$$\begin{pmatrix} \dot\sigma \\ \dot u \\ \dot\rho \end{pmatrix} + \begin{pmatrix} 0 & -d^* & 0 \\ d & 0 & d^* \\ 0 & -d & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \rho \end{pmatrix} = 0$$

strong

Find $\quad (\sigma, u, \rho) : [0,T] \to V^0 \times V^1 \times W^2 \quad$ s.t.

$$\begin{aligned} \langle \dot\sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, & \tau &\in V^0, \\ \langle \dot u, v \rangle + \langle d\sigma, v \rangle + \langle \rho, dv \rangle &= 0, & v &\in V^1, \\ \langle \dot\rho, \eta \rangle - \langle du, \eta \rangle &= 0, & \eta &\in W^2. \end{aligned}$$

weak

THEOREM

Given initial data $(\sigma_0, u_0, \rho_0) \in V^0 \times V^1 \times W^2$, ∃! solution $(\sigma, u, \rho) \in C^0([0,T], V^0 \times V^1 \times W^2) \cap C^1([0,T], W^0 \times W^1 \times W^2)$.

Proof: Uniqueness: $(\tau, v, \eta) = (\sigma, u, \rho)$. Existence: Hille–Yosida.

Example: Maxwell's equations

$$\dot{D} = \operatorname{curl} H \qquad \dot{B} = -\operatorname{curl} E$$
$$\operatorname{div} D = 0 \qquad \operatorname{div} B = 0$$
$$D = \epsilon E \qquad B = \mu H$$

$$W^0 = L^2(\Omega)$$
$$W^1 = L^2(\Omega, \mathbb{V}, \epsilon\, dx)$$
$$W^2 = L^2(\Omega, \mathbb{V}, \mu^{-1} dx)$$

$$W^0 \xrightarrow{\operatorname{grad}} W^1 \xrightarrow{-\operatorname{curl}} W^2$$

$(\sigma, E, B) : [0, T] \times \Omega \to \mathbb{R} \times \mathbb{V} \times \mathbb{V}$ solves

$$\langle \dot{\sigma}, \tau \rangle - \langle \epsilon E, \operatorname{grad} \tau \rangle = 0 \; \forall \tau,$$
$$\langle \epsilon \dot{E}, F \rangle + \langle \epsilon \operatorname{grad} \sigma, F \rangle - \langle \mu^{-1} B, \operatorname{curl} F \rangle = 0 \; \forall F,$$
$$\langle \mu^{-1} \dot{B}, C \rangle + \langle \mu^{-1} \operatorname{curl} E, C \rangle = 0 \; \forall C.$$

THEOREM

If σ, $\operatorname{div} \epsilon E$, and $\operatorname{div} B$ vanish for $t = 0$, then they vanish for all t, and E, B, $D = \epsilon E$, and $H = \mu^{-1} B$ satisfy Maxwell's equations.

Some other complexes

$\mathbb{R}^{3\times 3}$ symmetric

$$0 \to L^2 \otimes \mathbb{V} \xrightarrow{\text{sym grad}} L^2 \otimes \mathbb{S} \xrightarrow{\text{curl T curl}} L^2 \otimes \mathbb{S} \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \to 0$$

displacement — strain — stress — load

- $0 \to L^2 \otimes \mathbb{V} \xrightarrow{\text{sym grad}} L^2 \otimes \mathbb{S}$ primal method for elasticity
- $L^2 \otimes \mathbb{S} \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \to 0$ mixed method for elasticity

$\mathbb{R}^{3\times 3}$ trace-free

$$0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S} \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T} \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \to 0$$

- $0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}$ primal method for plate equation
- $L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S} \xrightarrow{\text{curl}} L^2 \otimes \mathbb{V}$ Einstein–Bianchi eqs (GR)