Finite Element Exterior Calculus and Applications

Part V

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De Rham complex

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

- $0 \to H^1(\Omega) \xrightarrow{\text{grad}} L^2(\Omega; \mathbb{R}^3)$:
  standard formulation of scalar Laplacian
- $H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} L^2(\Omega; \mathbb{R}^3)$:
  1-form Laplacian, Maxwell's equation based on E and $\sigma = \text{div}\, \epsilon E = 0$
- $H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega)$:
  2-form Laplacian, Maxwell's equation based on B and E
- $H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$:
  mixed formulation of scalar Laplacian

De Rham complex in 2D

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{rot}, \Omega) \xrightarrow{\text{rot}} L^2(\Omega) \to 0$$

or

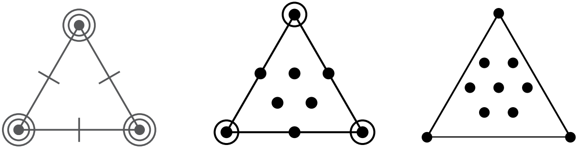
$$0 \to H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

- $0 \to H^1(\Omega) \xrightarrow{\text{curl}} L^2(\Omega; \mathbb{R}^2)$:
  standard formulation of scalar Laplacian
- $H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega)$:
  1-form Laplacian
- $H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$:
  mixed formulation of scalar Laplacian (Darcy flow)

Stokes complex in 2D and 3D

$$0 \to H^2(\Omega) \xrightarrow{\text{curl}} H^1(\Omega;\mathbb{R}^2) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Falk-Neilan shape fns: $\mathcal{P}_5\Lambda^0$ / $\mathcal{P}_4\Lambda^1$ / $\mathcal{P}_3\Lambda^2$



$$0 \to H^1(\Omega) \xrightarrow{\text{div}} H^1(\Omega,\text{curl};\mathbb{R}^3) \xrightarrow{\text{curl}} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

J. Evans '11

Elasticity complex

$$0 \to H^1(\Omega; R^3) \xrightarrow{\text{sym grad}} H(\text{curl}\, T\, \text{curl}, \Omega) \xrightarrow{\text{curl}\, T\, \text{curl}} H(\text{div}, \Omega; \mathcal{S}^{3\times 3}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \to$$

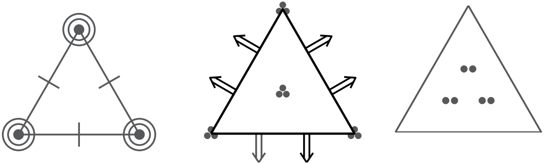
$$0 \to H^2(\Omega) \xrightarrow{\text{curl curl}} H(\text{div}, \Omega; \mathcal{S}^{2\times 2}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^2) \to 0$$

$$0 \to H^1(\Omega; \mathbb{R}^2) \xrightarrow{\text{sym grad}} H(\text{rot rot}, \Omega; \mathcal{S}^{2\times 2}) \xrightarrow{\text{rot rot}} L^2(\Omega) \to 0$$

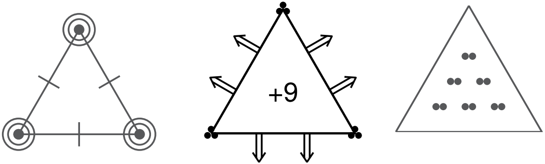
- $0 \to H^1(\Omega; \mathbb{R}^2) \xrightarrow{\text{sym grad}} H(\text{rot rot}, \Omega; \mathcal{S}^{2\times 2})$:
  displacement formulation of elasticity
- $H((\text{div}, \Omega; \mathcal{S}^{2\times 2}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^2) \to 0$:
  mixed formulation of elasticity (strong symmetry)

Mixed elasticity elements (2D strong symmetry)

$$0 \to H^2(\Omega) \xrightarrow{\text{curl curl}} H(\text{div}, \Omega; \mathcal{S}^{2\times 2}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^2) \to 0$$



AW 2002



Hu Jun-Shangyou Zhang 2015

New complexes from old: a simple case

Suppose $0 \to \bar{W}^1 \xrightarrow{\bar{d},\bar{V}^1} \bar{W}^2$ and $0 \to \tilde{W}^1 \xrightarrow{\tilde{d},\tilde{V}^1} \tilde{W}^2$ are closed Hilbert complexes, and that there is a bounded linear isomorphism $S : \tilde{W}^1 \to \bar{W}^2$.

$$\begin{array}{ccccc} 0 & \longrightarrow & \bar{W}^1 & \xrightarrow{\bar{d},\,\bar{V}^1} & \bar{W}^2 \\ & & & \nearrow^{S} & \\ 0 & \longrightarrow & \tilde{W}^1 & \xrightarrow{\tilde{d},\,\tilde{V}^1} & \tilde{W}^2 \end{array}$$

We define a new short Hilbert complex:

- $V^1 = \{(u,\phi) \in \bar{V}^1 \times \tilde{V}^1 \mid du = S\phi\}$
- W^1 is the competion of V^1 wrt the norm $\|u,\phi\|_W := \|u\|_{\bar{W}}$ (S is injective)
- $W^2 = \tilde{W}^2$
- $d : V^1 \subset W^1 \to W^2$ is given by $d(u,\phi) = \tilde{d}\phi$.

THEOREM

Suppose that the initial two H-complexes are closed and exact. Then

$0 \longrightarrow W^1 \xrightarrow{d,V^1} W^2$ *is also a closed, exact H-complex.*

The Hodge Laplacian for the derived complex

$$\begin{array}{ccccc} 0 & \longrightarrow & \bar W^1 & \xrightarrow{\bar d,\,\bar V^1} & \bar W^2 \\ & & & \nearrow_{S} & \\ 0 & \longrightarrow & \tilde W^1 & \xrightarrow{\tilde d,\,\tilde V^1} & \tilde W^2 \end{array} \qquad 0 \longrightarrow W^1 \xrightarrow{d,V^1} W^2$$

Hodge Lap: Find $(u,\phi) \in V^1$ st $\quad \langle \tilde d\phi, \tilde d\psi\rangle = \langle f, v\rangle, \quad (v,\psi) \in V^1$

$$\begin{aligned} V^1 &= \{(u,\phi) \in \bar V^1 \times \tilde V^1 \,|\, du = S\phi\} \\ &= \{(u,\phi) \in \bar V^1 \times \tilde V^1 \,|\, \langle du - S\phi, \mu\rangle = 0 \;\forall \mu \in \bar W^2\} \end{aligned}$$

Implement via Lagrange multiplier: Find $u \in \bar V^1, \phi \in \tilde V^1, \lambda \in \bar W^2$ s.t.

$$\begin{aligned} \langle \tilde d\phi, \tilde d\psi\rangle + \langle \lambda, dv - S\psi\rangle &= \langle f, v\rangle, & v \in \bar V^1,\ \psi \in \tilde V^1, \\ \langle du - S\phi, \mu\rangle &= 0, & \mu \in \bar W^2 \end{aligned}$$

New norm on $\bar W^2$: $\quad |||\mu||| = \sup_{v \in \bar V^1, \psi \in \tilde V^1} \dfrac{\langle \mu, dv - S\psi\rangle}{\|v\|_V + \|\psi\|_{\tilde V}} \quad$ ($S\tilde V^1$ is dense)

This mixed method satisfies the Brezzi conditions and so is well-posed.

The idea is to mimic the construction on the discrete level. Choose two discrete subcomplexes which admit commuting projections:

$$0 \longrightarrow \bar{V}_h^1 \xrightarrow{d} \bar{V}_h^2 \qquad 0 \longrightarrow \tilde{V}_h^1 \xrightarrow{\tilde{d}} \tilde{V}_h^2$$

Create a discrete connection map as $S_h = \bar{\Pi}_h^2 S : \tilde{V}_h^1 \to \bar{V}_h^2$ where $\bar{\Pi}_h^2$ is the canonical projection. This gives the mixed method:

Find $u_h \in \bar{V}_h^1$, $\phi_h \in \tilde{V}_h^1$, $\lambda_h \in \bar{V}_h^2$ s.t.

$$\begin{aligned} \langle \tilde{d}\phi_h, \tilde{d}\psi \rangle + \langle \lambda_h, dv - \bar{\Pi}_h^2 S\psi \rangle &= \langle f, v \rangle, \quad && v \in \bar{V}_h^1,\ \psi \in \tilde{V}_h^1, \\ \langle du_h - \bar{\Pi}_h^2 S\phi_h, \mu \rangle &= 0, && \mu \in \bar{V}_h^2 \end{aligned}$$

We make the *surjectivity assumption* $\bar{\Pi}_h^2 S \tilde{\Pi}_h^1 = \bar{\Pi}_h^2$. We can then prove that the mixed method is convergent.

Example: the biharmonic

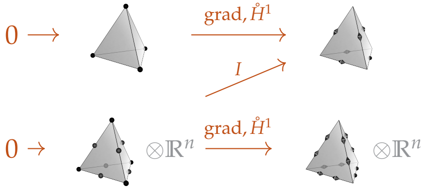
$$\begin{array}{ccccc} 0 & \longrightarrow & L^2 & \xrightarrow{\operatorname{grad},\,\mathring{H}^1} & L^2(\Omega;\mathbb{R}^n) \\ & & & \nearrow I & \\ 0 & \longrightarrow & L^2(\Omega;\mathbb{R}^n) & \xrightarrow{\operatorname{grad},\,\mathring{H}^1} & L^2(\Omega;\mathbb{R}^{n\times n}) \end{array}$$

$$\begin{aligned} V^1 &= \{(u,\phi) \in \mathring{H}^1(\Omega) \times \mathring{H}^1(\Omega;\mathbb{R}^n) \mid \phi = \operatorname{grad} u\} \\ &= \{(u, \operatorname{grad} u) \mid u \in \mathring{H}^2(\Omega)\} \\ &\cong \mathring{H}^2 \end{aligned}$$

$$\begin{array}{ccccc} 0 & \longrightarrow & W^1 & \xrightarrow{d,\,V^1} & W^2 \\ & & & & \\ 0 & \longrightarrow & L^2 & \xrightarrow{\operatorname{grad}\operatorname{grad},\,\mathring{H}^2} & L^2(\Omega;\mathbb{R}^{n\times n}) \end{array}$$

FEEC discretization of the biharmonic

$$
\begin{array}{l}
0 \longrightarrow L^2 \xrightarrow{\text{grad},\, \mathring{H}^1} L^2(\Omega;\mathbb{R}^n) \\
\qquad\qquad\qquad \nearrow^{I} \\
0 \to L^2(\Omega;\mathbb{R}^n) \xrightarrow{\text{grad},\, \mathring{H}^1} L^2(\Omega;\mathbb{R}^{n\times n})
\end{array}
$$



This gives a family of mixed methods for the biharmonic based on a different formulation than the classical methods (Ciarlet–Raviart, Hellan–Herman–Johnson, . . .). It is related (in 2D) to the approach of Durán–Liberman for the Reissner–Mindlin plate.

Elasticity with weak symmetry

The mixed formulation of elasticity with *weak symmetry* is more amenable to discretization than the standard mixed formulation.

Fraeijs de Veubeke '75

$$p = \operatorname{skw} \operatorname{grad} u, \quad A\sigma = \operatorname{grad} u - p$$

Find $\quad \sigma \in L^2(\Omega; \mathbb{R}^{n\times n}), u \in L^2(\Omega; \mathbb{R}^n), p \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \quad$ s.t.

$$\begin{aligned} \langle A\sigma, \tau\rangle + \langle u, \operatorname{div} \tau\rangle + \langle p, \tau\rangle &= 0, && \tau \in L^2(\Omega; \mathbb{R}^{n\times n}) \\ -\langle \operatorname{div} \sigma, v\rangle &= \langle f, v\rangle, && v \in L^2(\Omega; \mathbb{R}^n) \\ -\langle \sigma, q\rangle &= 0, && q \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \end{aligned}$$

This is exactly the mixed Hodge Laplacian for the complex

$$L^2_A(\Omega; \mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} L^2(\Omega; \mathbb{R}^n) \oplus L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

supposing that it is exact.

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

To show the complex is exactness, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{array}{ccccccc}
 & & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\text{skw}}) & \xrightarrow{\operatorname{div}} & L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) & \longrightarrow & 0 \\
 & \overset{S}{\nearrow} & & \overset{-\operatorname{skw}}{\nearrow} & & & \\
L^2(\Omega;\mathbb{R}^{n\times n}) & \xrightarrow{\operatorname{curl}} & L^2(\Omega;\mathbb{R}^{n\times n}) & \xrightarrow{-\operatorname{div}} & L^2(\Omega;\mathbb{R}^n) & \longrightarrow & 0
\end{array}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}) \longrightarrow 0$$

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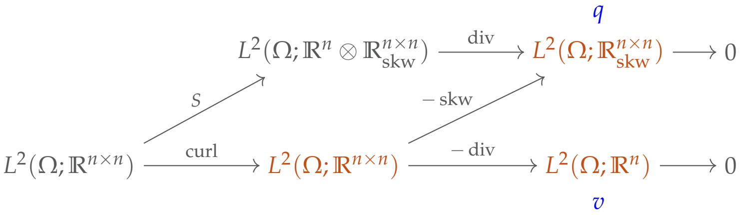

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

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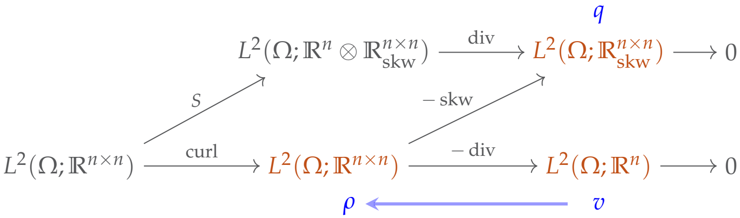

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}) \longrightarrow 0$$

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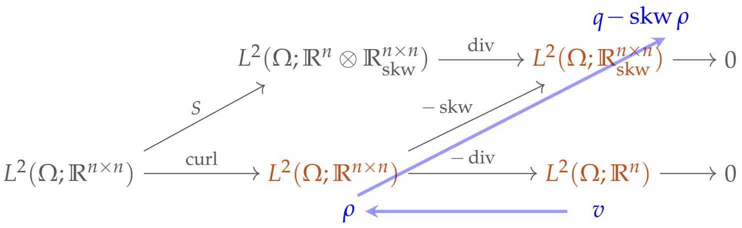
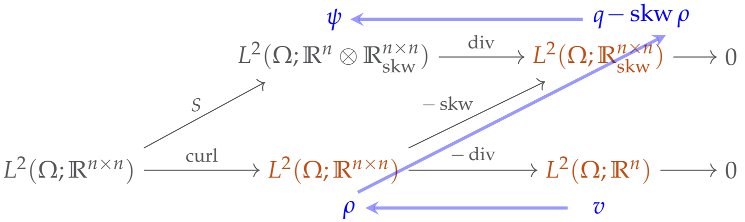


$S\tau=\tau^T-\operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

To show the complex is exactness, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

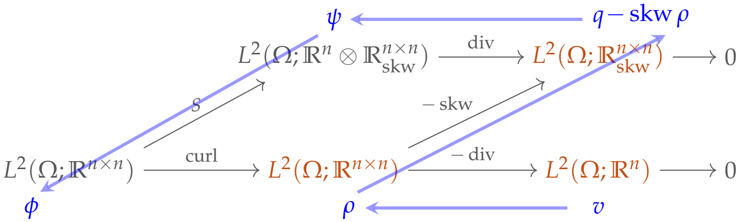


$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

To show the complex is exactness, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

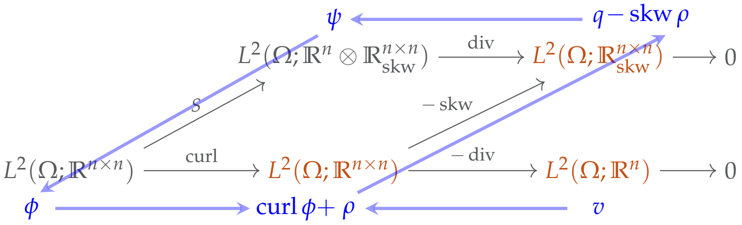


$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \longrightarrow 0$$

To show the complex is exactness, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:



$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Discretization

To discretize we select discrete de Rham subcomplexes with commuting projs

$$\bar V_h^1 \xrightarrow{\operatorname{div}} \bar V_h^2 \to 0, \qquad \tilde V_h^0 \xrightarrow{\operatorname{curl}} \tilde V_h^1 \xrightarrow{-\operatorname{div}} \tilde V_h^2 \to 0$$

to get the discrete complex

$$\tilde V_h^1 \otimes \mathbb{R}^n \xrightarrow{(-\operatorname{div}, -\bar\pi_h^2 \operatorname{skw})} (\tilde V_h^2 \otimes \mathbb{R}^n) \times (\bar V_h^2 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}) \to 0$$

We get stability if we can carry out the diagram chase on:

$$\begin{array}{ccccccc}
 & & \bar V_h^1 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} & \xrightarrow{\operatorname{div}} & \bar V_h^2 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} & \to & 0 \\
 & \nearrow^{\bar\pi_h^1 S} & & \nearrow^{-\bar\pi_h^2 \operatorname{skw}} & & & \\
\tilde V_h^0 \otimes \mathbb{R}^n & \xrightarrow{\operatorname{curl}} & \tilde V_h^1 \otimes \mathbb{R}^n & \xrightarrow{-\operatorname{div}} & \tilde V_h^2 \otimes \mathbb{R}^n & \longrightarrow & 0
\end{array}$$

This requires that $\bar\pi_h^1 S : \tilde V_h^0 \otimes \mathbb{R}^n \to \bar V_h^1 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}$ is *surjective*.

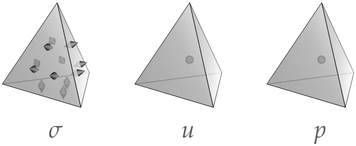
Stable elements

The requirement that $\bar{\pi}_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to \bar{V}_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}}$ is surjective can be checked by looking at DOFs.

The simplest choice is

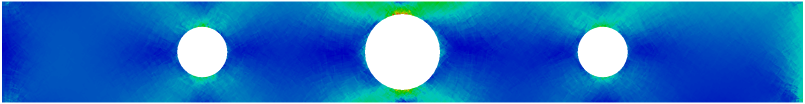
$$\mathcal{P}_r^- \Lambda^{n-1} \xrightarrow{\text{div}} \mathcal{P}_r^- \Lambda^n \to 0, \quad \mathcal{P}_{r+1}^- \Lambda^{n-2} \xrightarrow{\text{curl}} \mathcal{P}_r \Lambda^{n-1} \xrightarrow{-\text{div}} \mathcal{P}_{r-1} \Lambda^n \to 0$$

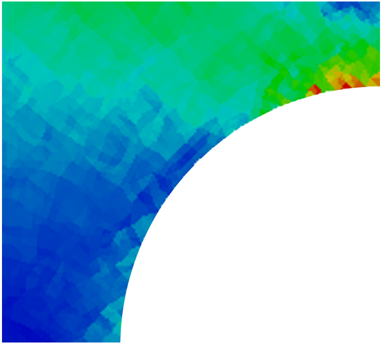
This gives the elements of DNA–Falk–Winther ’07



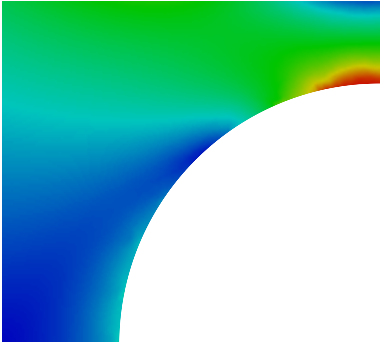
Other elements:
Cockburn–Gopalakrishnan–Guzmán,
Gopalakrishnan–Guzmán, Stenberg, …

Nearly incompressible material





displacement



mixed

$$\text{Riem} = \text{Ricci} + \text{Weyl}$$

$$\text{Weyl} = (C_{abcd}) = \begin{pmatrix} E & B \\ B & -E \end{pmatrix}$$

E, B 3×3 symmetric, traceless

Einstein equations + Bianchi identity $\implies$ Einstein–Bianchi eqs:

Find: $E, B : [0, T] \to \mathcal{S}^{3\times 3}$ such that

$$\begin{aligned} \dot{E} &= -\operatorname{curl} B, & \dot{B} &= \operatorname{curl} E, \\ \operatorname{div} E &= 0, & \operatorname{div} B &= 0, \\ \operatorname{tr} E &= 0, & \operatorname{tr} B &= 0. \end{aligned}$$

Einstein–Bianchi as an abstract Hodge wave equation

$$L^2(\Omega) \xrightarrow{\text{grad grad}, H^2} L^2(\Omega; \mathcal{S}) \xrightarrow{\text{curl}, H(\text{curl})} L^2(\Omega; \mathbb{T})$$

Find $\quad (\sigma, E, B) : [0, T] \to H^2 \times H(\text{curl}; \mathcal{S}) \times L^2(\Omega; \mathbb{T}) \quad$ s.t.

$$\begin{aligned}
&\langle \dot\sigma, \tau \rangle - \langle u, \text{grad grad}\, \tau \rangle = 0, && \tau \in H^2, \\
&\langle \dot E, F \rangle + \langle \text{grad grad}\, \sigma, F \rangle + \langle B, \text{curl}\, F \rangle = 0, && F \in H(\text{curl}; \mathcal{S}), \\
&\langle \dot B, C \rangle - \langle \text{curl}\, E, C \rangle = 0, && C \in L^2(\Omega; \mathbb{T}).
\end{aligned}$$

$$\dot\sigma = \text{div div}\, E, \quad \dot E = -\,\text{grad grad}\, \sigma - \text{sym curl}\, B, \quad \dot B = \text{curl}\, E$$

THEOREM

Suppose $\sigma(0) = 0$ and $E(0)$ and $B(0)$ are TSD. Then $\sigma = 0$ and E and B are TSD for all time, and E and B satisfy the linearized EB equations.

Obstacles to discretization

To proceed we need finite element subspaces which form a subcomplex with bounded cochain projections. There are two serious obstacles.

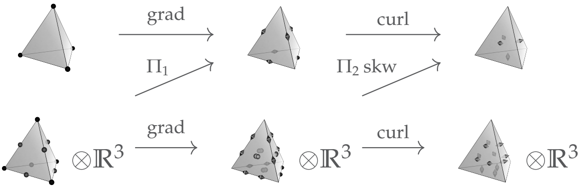
1. It is difficult to create a finite element subspace of H^2 because of the second derivatives.
2. It is difficult to create a finite element subspace of $H(\mathrm{curl}; \mathcal{S})$ because of the symmetry.

For each of these obstacles we are guided by their solution in simpler context (biharmonic, elasticity).

Combining these ideas leads to a first order formulation of EB using six variables.

$$\begin{array}{ccccc} L^2(\Omega) & \xrightarrow{\text{grad}} & L^2(\Omega;\mathbb{R}^3) & \xrightarrow{\text{curl}} & L^2(\Omega;\mathbb{R}^3) \\ & \nearrow^{I} & & \nearrow^{\text{skw}} & \\ L^2(\Omega;\mathbb{R}^3) & \xrightarrow{\text{grad}} & L^2(\Omega;\mathbb{R}^{3\times 3}) & \xrightarrow{\text{curl}} & L^2(\Omega;\mathbb{R}^{3\times 3}) \\ & & E & & B \end{array}$$

FEEC guides us to an appropriate choice of elements.



Which complexes can we construct from the de Rham complex?

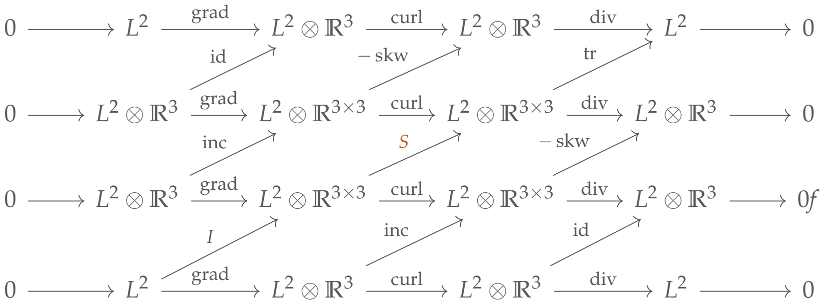


Diagram commutes. Diagonal maps are isomorphisms, subdiagonal injections, superdiagonal surjections.