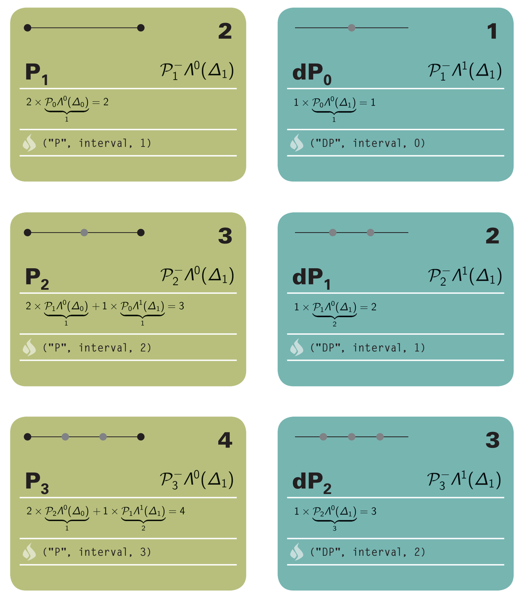
Periodic Table of the Finite Elements



$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is

$$\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1},$$

where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

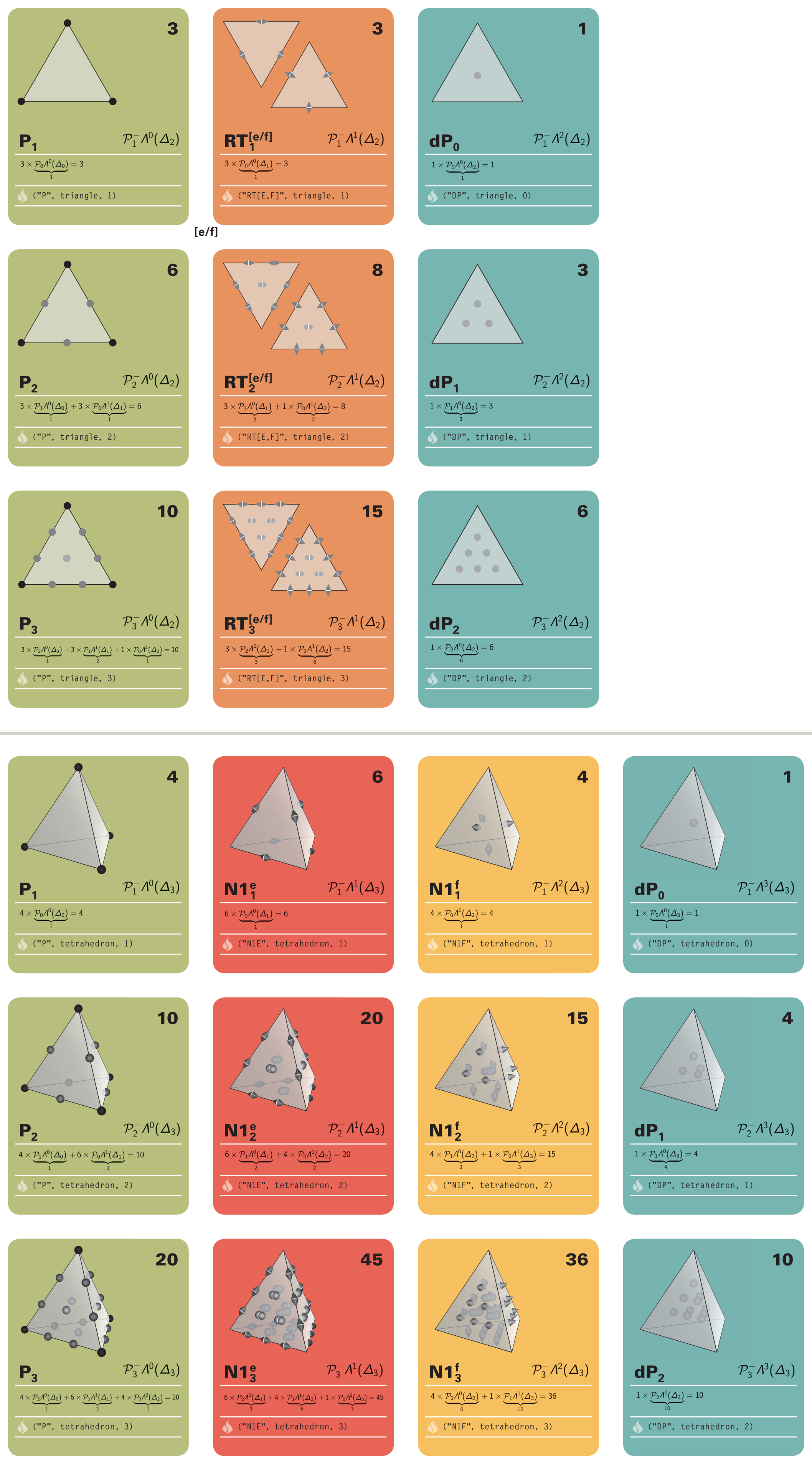
$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

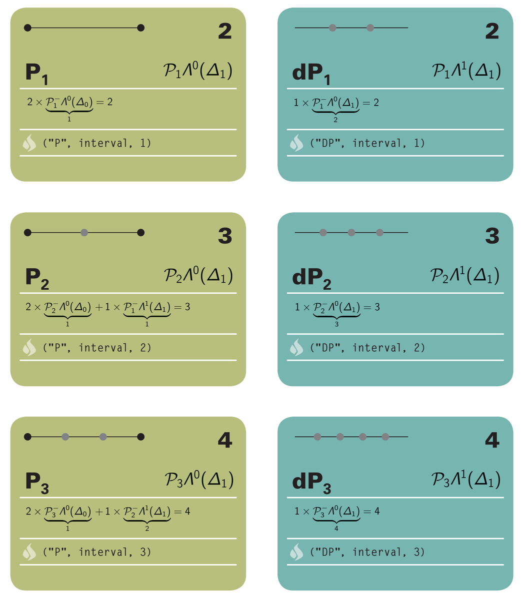
The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^n.$$





$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension

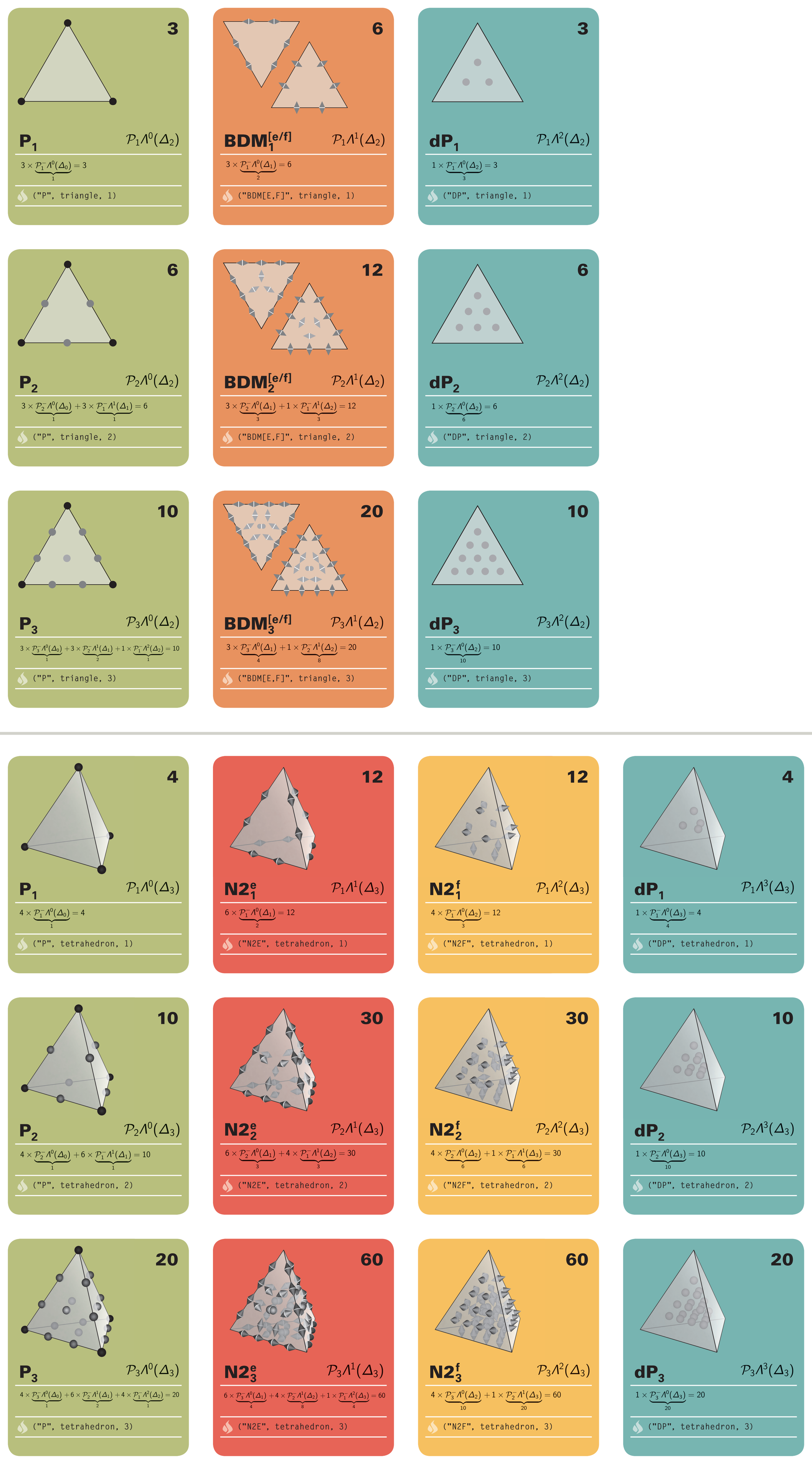
$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

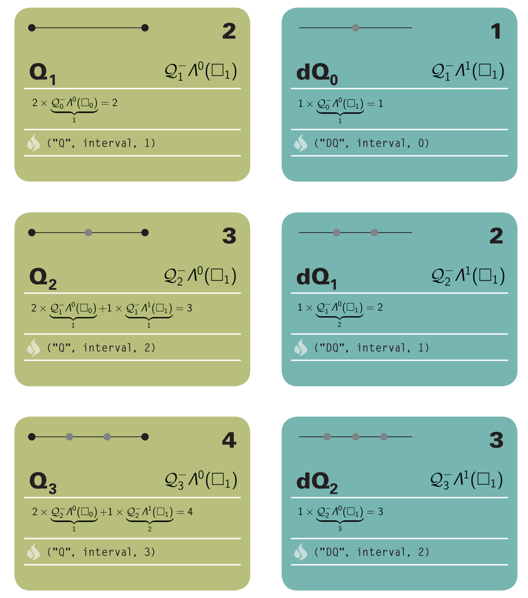
The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a $\mathcal{P}_r^-$ space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{P}_r\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_{r-n}\Lambda^n.$$





$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\square_n = I^n$ is given by

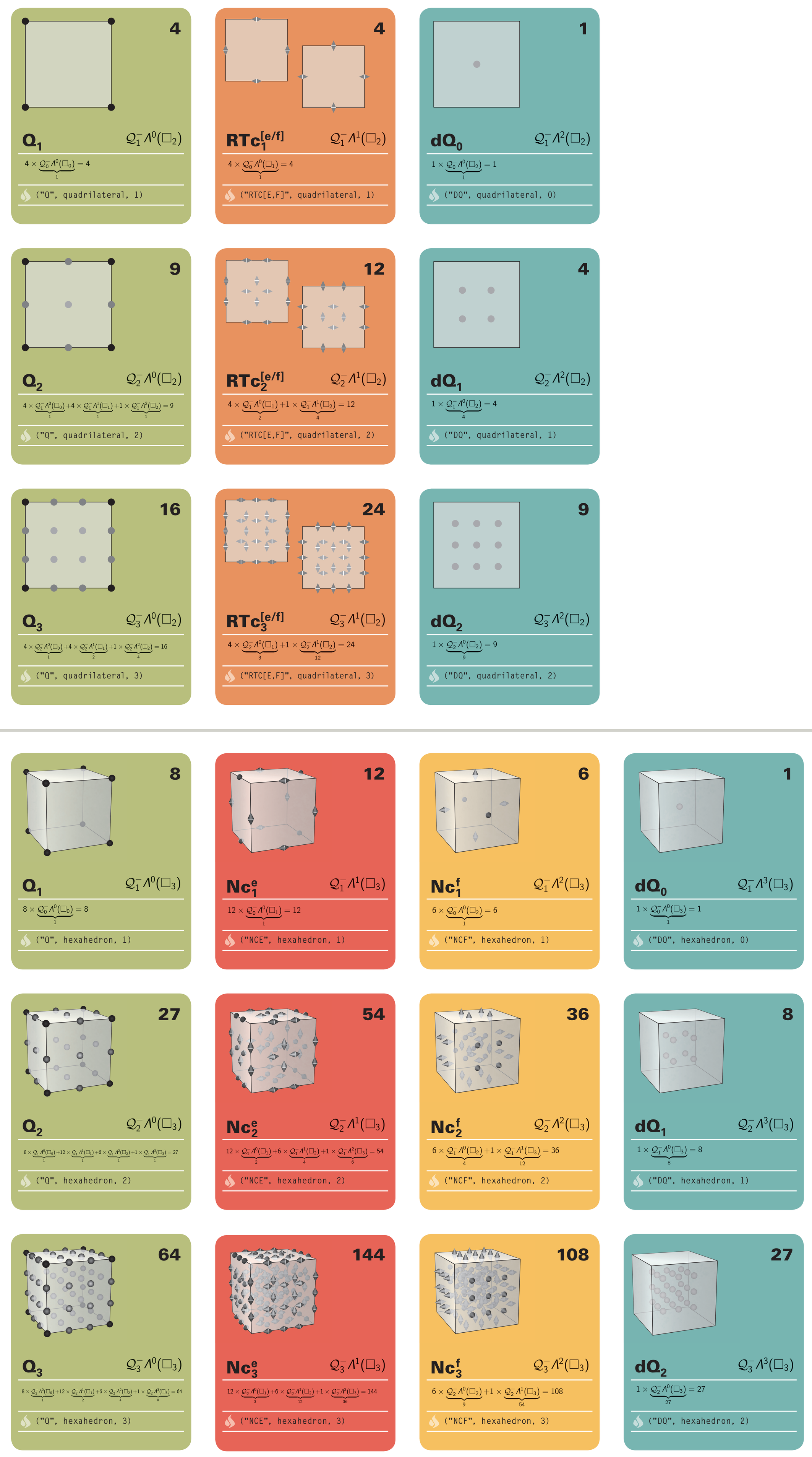
$$\bigoplus_{\sigma\in\Sigma(k,n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k},$$

where $\Sigma(k,n)$ denotes the increasing maps $\{1,\ldots,k\} \to \{1,\ldots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k (r+1)^{n-k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{Q}^-_{r-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{Q}_r^-\Lambda^n.$$



$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by

$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\geq 1}[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1} \oplus \mathrm{d}\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k],$$

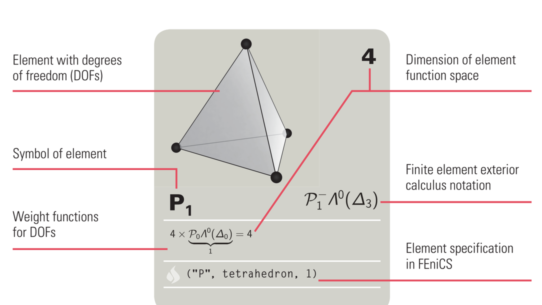
where $\mathcal{H}_{r,l}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d\geq k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{S}_r\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{S}_{r-n}\Lambda^n.$$

Legend



$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7
n = 2	0	3	6	10	15	21	28	36
	1	3	8	15	24	35	48	63
	2	1	3	6	10	15	21	28
n = 3	0	4	10	20	35	56	84	120
	1	6	20	45	84	140	216	315
	2	4	15	36	70	120	189	280
	3	1	4	10	20	35	56	84
n = 4	0	5	15	35	70	126	210	330
	1	10	40	105	224	420	720	1155
	2	10	45	126	280	540	945	1540
	3	5	24	70	160	315	560	924
	4	1	5	15	35	70	126	210

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
n = 2	0	3	6	10	15	21	28	36
	1	6	12	20	30	42	56	72
	2	3	6	10	15	21	28	36
n = 3	0	4	10	20	35	56	84	120
	1	12	30	60	105	168	252	360
	2	12	30	60	105	168	252	360
	3	4	10	20	35	56	84	120
n = 4	0	5	15	35	70	126	210	330
	1	20	60	140	280	504	840	1320
	2	30	90	210	420	756	1260	1980
	3	20	60	140	280	504	840	1320
	4	5	15	35	70	126	210	330

$$\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k(r+1)^{n-k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7
n = 2	0	4	9	16	25	36	49	64
	1	4	12	24	40	60	84	112
	2	1	4	9	16	25	36	49
n = 3	0	8	27	64	125	216	343	512
	1	12	54	144	300	540	882	1344
	2	6	36	108	240	450	756	1176
	3	1	8	27	64	125	216	343
n = 4	0	16	81	256	625	1296	2401	4096
	1	32	216	768	2000	4320	8232	14336
	2	24	216	864	2400	5400	10584	18816
	3	8	96	432	1280	3000	6048	10976
	4	1	16	81	256	625	1296	2401

$$\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d=k}^{\min(n,\lfloor r/2\rfloor+k)} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
n = 2	0	4	8	12	17	23	30	38
	1	8	14	22	32	44	58	74
	2	3	6	10	15	21	28	36
n = 3	0	8	20	32	50	74	105	144
	1	24	48	84	135	204	294	408
	2	18	39	72	120	186	273	384
	3	4	10	20	35	56	84	120
n = 4	0	16	48	80	136	216	328	480
	1	64	144	272	472	768	1188	1764
	2	72	168	336	606	1014	1602	2418
	3	32	84	180	340	588	952	1464
	4	5	15	35	70	126	210	330

Finite elements

The table presents the primary spaces of finite elements for the discretization of the fundamental operators of vector calculus: the gradient, curl, and divergence. A finite element space is a space of piecewise polynomial functions on a domain determined by: (1) a mesh of the domain into polyhedral cells called *elements*, (2) a finite dimensional space of polynomial functions on each element called the *shape functions*, and (3) a unisolvent set of functionals on the shape functions of each element called *degrees of freedom* (DOFs), each DOF being associated to a (generalized) face of the element, and specifying a quantity which takes a single value for all elements sharing the face. The element diagrams depict the DOFs and their association to faces.

The spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ depicted on the left half of the table are the two primary families of finite element spaces for meshes of simplices, and the spaces $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ on the right side are for meshes of cubes or boxes. Each is defined in any dimension $n \geq 1$ for each value of the polynomial degree $r \geq 1$, and each value of $0 \leq k \leq n$. The parameter k refers to the operator: the spaces consist of differential k-forms which belong to the domain of the kth exterior derivative. Thus for $k = 0$, the spaces discretize the Sobolev space H^1, the domain of the gradient operator; for $k = 1$, they discretize $H(\mathrm{curl})$, the domain of the curl; for $k = n-1$ they discretize $H(\mathrm{div})$, the domain of the divergence; and for $k = n$, they discretize L^2.

The spaces $\mathcal{P}_r^-\Lambda^0$ and $\mathcal{P}_r\Lambda^0$, which coincide, are the earliest finite elements, going back in the case $r = 1$ of linear elements to *Courant*,[1] and collectively referred to as *the Lagrange elements*. The spaces $\mathcal{P}_{r+1}^-\Lambda^n$ and $\mathcal{P}_r\Lambda^n$, which also coincide, are the discontinuous *Galerkin* elements, consisting of piecewise polynomials with no interelement continuity imposed, first introduced by *Reed and Hill*.[2] The space $\mathcal{P}_r^-\Lambda^1$ in 2 dimensions was introduced by *Raviart and Thomas*[3] and generalized to the 3-dimensional spaces $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r^-\Lambda^2$ by *Nédélec*,[4] while $\mathcal{P}_r\Lambda^1$ is due to *Brezzi, Douglas and Marini*[5] in 2 dimensions, its generalization to 3 dimensions again due to *Nédélec*.[6] The unified treatment and notation of the $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ families is due to *Arnold, Falk and Winther* as part of *finite element exterior calculus*,[7] extending earlier work of *Hiptmair* for the $\mathcal{P}_r^-\Lambda^k$ family.[8] The space $\mathcal{P}_1^-\Lambda^k$ is the span of the *elementary forms* introduced by *Whitney*.[9]

The family $\mathcal{Q}_r^-\Lambda^k$ of cubical elements can be derived from the 1-dimensional Lagrange and discontinuous Galerkin elements by a tensor product construction detailed by *Arnold, Boffi and Bonizzoni*,[10] but for the most part were presented individually along with the corresponding simplicial elements in the papers mentioned. The second cubical family $\mathcal{S}_r\Lambda^k$ is due to *Arnold and Awanou*.[11]

The finite elements in this table have been implemented as part of the *FEniCS Project*.[12,13,14] Each may be referenced in the *Unified Form Language* (UFL)[15] by giving its family, shape, and degree, with the family as shown on the table. For example, the space $\mathcal{P}_3\Lambda^1(\Delta_3)$ may be referred to in UFL as:

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FiniteElement("N2E", tetrahedron, 3)
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Alternatively, the elements may be accessed in a uniform fashion as:

```
FiniteElement("P-", shape, r, k)
FiniteElement("P",  shape, r, k)
FiniteElement("Q-", shape, r, k)
FiniteElement("S",  shape, r, k)
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for $\mathcal{P}_r^-\Lambda^k$, $\mathcal{P}_r\Lambda^k$, $\mathcal{Q}_r^-\Lambda^k$, and $\mathcal{S}_r\Lambda^k$, respectively.

References

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Concept and scientific content: Douglas N. Arnold (University of Minnesota) and Anders Logg (Chalmers University of Technology). Graphic design: Mattias Schläger. The production of this poster has been supported by Simula Research Laboratory and is partially based on work supported by the U.S. National Science Foundation under grant DMS-1115291. Findings do not necessarily represent the views of Simula or of the NSF.

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