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INTERIOR ESTIMATES FOR SOME NONCONFORMING AND MIXED FINITE ELEMENT METHODS

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CHAPTER 1

INTRODUCTION

Finite element methods are widely used for approximating elliptic boundary value problems. Usually the accuracy of such numerical methods depend on both the smoothness of the exact solution and on the order of complete polynomials in the finite element space. To be specific, consider the Dirichlet problem for the Poisson equation

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ (so that the finite element space can be constructed without error in approximating the boundary) and f is some given function. The standard finite element method for (1.1) consists of constructing a one-parameter family of continuous piecewise polynomials subspaces $\mathring{V}_h$ of the Hilbert space $\mathring{H}^1(\Omega)$ and using the Ritz-Galerkin method to compute an approximation $u_h \in \mathring{V}_h$. The standard error estimate gives

$$\|u - u_h\|_{1,\Omega} \le C \inf_{v \in \mathring{V}_h} \|u - v\|_{1,\Omega} \le C h^{\min(k,r-1)} \|u\|_{r,\Omega}, \tag{1.2}$$

where $\|\cdot\|_{r,\Omega}$ is the norm on Hilbert space $H^r(\Omega)$ and k is the order of complete polynomials in the finite element space $\mathring{V}_h$. In order for the finite element solution to achieve the optimal convergence rate, the exact solution u must be sufficiently regular. Namely, if $r \ge k+1$, then (1.2) will result in an $O(h^k)$ order convergence rate, which is best possible for the degree of polynomials used. Otherwise a loss of accuracy will occur.

In practice, it often happens that $r < k+1$. For example, when Ω is a nonconvex polygon, the exact solution will generally have *corner singularities*, and one cannot expect u to be in $H^2(\Omega)$. So no matter how high the order of the finite element space $\mathring{V}_h$ is, the finite element approximation does not even achieve first order convergence. The situation is even worse for the plane elasticity problem, described by a second order vector elliptic equation. In this case, the solution may not be in $H^2(\Omega)$ even if Ω is a convex polygon [29] (under some boundary conditions). We also note that there are other important situations when the exact solution is singular or nearly so, even when the boundary is smooth, for example, in singular perturbation problems or problems with concentrated loads.

In the examples mentioned above, the exact solution is smooth in a large part of the domain and the singularity is a local phenomenon. Therefore, it is natural to ask whether u_h approximates u better where u is smoother. Interior error estimates address this question.

Interior error estimates for finite element discretizations were first introduced by Nitsche and Schatz [33] for second order scalar elliptic equations in 1974. They proved that for h sufficiently small

$$\|u - u_h\|_{1,\Omega_0} \leq C\big(\inf_{v \in \mathring{V}_h} \|u - v\|_{1,\Omega_1} + \|u - u_h\|_{-p,\Omega_1} \big), \tag{1.3}$$

for $\Omega_0 \Subset \Omega_1 \Subset \Omega$ (here $A \Subset B$ means that $\bar{A} \subset B$) and any nonnegative integer p. Here C is a constant that is independent of u, u_h, and h. This estimate says that the local accuracy of the finite element approximation is bounded in terms of two factors: the local approximability of the exact solution by the finite element space and the global approximability measured in an arbitrarily weak Sobolev norm on a slightly larger domain. The usual way to estimate $\|u - u_h\|_{-p,\Omega_1}$ is to use the fact that $\|u - u_h\|_{-p,\Omega_1} \leq \|u - u_h\|_{-p,\Omega}$, for which the estimate is available by using Nitsche's duality technique. The significance of the negative norm is that, under some very important circumstances, one can prove higher rates of convergence in the negative norm than that in the energy norm. Therefore, better convergence rates may be obtained in the interior domain. But it does not imply that one can always recover the optimal convergence rate. For example, as a direct application of (1.3) and the standard convergence theory of the finite element method, it is easy to see that if linear Lagrange elements are used for the Poisson equation on an L-shaped domain with a smooth forcing function, then $\|u - u_h\|_{1,\Omega_0}$ is of $O(h)$ for any interior region Ω_0. However, if quadratic Lagrange elements are used for the same problem, $\|u - u_h\|_{1,\Omega_0}$ is only of order $O(h^{4/3})$, which is less than the optimal $O(h^2)$ rate (but better than the $O(h^{2/3})$ global rate). This phenomenon is called the *pollution effect* of the boundary singularity.

In 1977, Schatz and Wahlbin extended the idea of [33] and established interior estimates in the maximum norm [35] for second order elliptic equations. They proved that

$$\|u - u_h\|_{\infty,\Omega_0} \leq C\big((\ln \frac{1}{h})^{\bar{r}} \inf_{v \in \mathring{V}_h} \|u - v\|_{\infty,\Omega_1} + \|u - u_h\|_{-p,\Omega_1} \big), \tag{1.4}$$

where $\|\cdot\|_{\infty,\Omega_0}$ represents the usual maximum norm and $\bar{r} = 1$ for linear elements in $\mathbb{R}^2$, $\bar{r} = 0$, otherwise. This was later generalized to allow Ω_0 to intersect the boundary of Ω.

Interior error estimates are important for other reasons as well. In some cases, mesh refinement and post-processing schemes to improve the initial approximation can be designed by using the information obtained from a local analysis. In 1979 Schatz and Wahlbin [37], based on (1.4), gave a systematic mesh refinement procedure for the finite element method for second order elliptic equations on polygonal domains and showed that optimal global convergence rates could be obtained. In 1983, they studied in detail the approximation of the standard finite element method for the singular perturbed second order elliptic equation, where a strong boundary layer effect exists [38], again utilizing the interior convergence theory. In 1985 Eriksson [24], [25] applied the local analysis method to the second order elliptic equations with singular forcing functions and designed an adaptive mesh refinement scheme to obtain optimal convergence rates. They also generalized such methods to some time dependent problems [27].

Interior error estimates have also been used successfully to study a posteriori estimators. In 1988 Eriksson and Johnson [26] introduced two a posteriori error estimators based on local difference quotients of the numerical solution. Their analysis was based on the interior convergence theory in [33]. Zhu and Zienkiewicz [45], [46] proposed several adaptive procedures for finite element methods based on smoothing techniques. In 1991, Babuška and Rodríguez [9] gave a complete study of these estimators by using the interior estimate results of Bramble and Schatz [12]. In 1992, Durán [22], [23] proved the asymptotic exactness of several a posteriori error estimators by Bank and Weisser [10] by applying the interior superconvergence results of Whiteman and Wheeler [42].

The interior convergence theory is reasonably well understood for standard finite element methods. For a comprehensive review, see [41]. But there are only few results in this area for mixed finite element methods. The difficulty in obtaining interior estimates for mixed methods can be understood by considering how an interior estimate is usually obtained: first the exact solution is restricted to a local domain and its projection is constructed; then the difference between the global finite element solution and the local projection of the exact solution is estimated via duality and energy arguments. For the interior analysis of a mixed method, there are two new aspects compared to that for a standard one: the coupling of local projections and the balancing of two different norms. The resolution of these problems depends on the specific mixed formulation. In 1985 Douglas and Milner [20] adapted the

Nitsche-Schatz approach to the Raviart-Thomas mixed method for scalar second order elliptic problems. Their work took advantage of the so–called "commuting diagram property" [21] between the two discrete spaces. Recently, Gastaldi [28] obtained interior error estimates for some finite element methods for the Reissner–Mindlin plate model. Her work is similar in spirit to that of Chapter 4. However it is for the Brezzi-Bathe-Fortin family of elements for the Reissner-Mindlin plate [14], for which the variational formulation is different. The "commuting diagram property" plays an important role in Gastaldi's work, but does not enter here.

In this thesis we establish interior estimates for some nonconforming and mixed finite element methods. Our primary goal is the interior error analysis for the the Arnold-Falk element for the Reissner-Mindlin plate model [3]. Via the Helmholtz decomposition, the Reissner-Mindlin system can be transformed into an uncoupled system of two Poisson equations and a singularly perturbed variant of the Stokes system. Using a discrete Helmholtz decomposition theorem, the Arnold-Falk element can be viewed as combination of nonconforming linear elements for the Poisson equations and the MINI element [2] for the Stokes-like system. Therefore the interior analysis of the Arnold-Falk element requires analysis of the nonconforming piecewise linear finite element for the Poisson equation and of the MINI element for the Stokes-like system, and so we consider those problems, which are also of interest in their own right, first.

The thesis is organized as follows. Chapter 2 defines with some notation and derives interior estimates for the linear nonconforming finite element method for the Poisson equation. This result will be used later in Chapter 4 in the interior estimate of the Arnold-Falk element for the Reissner-Mindlin plate model. Because of the relative simplicity of this chapter, it also serves to review the standard procedure for obtaining interior error estimates. Chapter 3 gives interior error estimates in the energy norm for a wide class of finite element methods for the Stokes equations. In Chapter 4 we study the interior error estimate of the Arnold-Falk element for the Reissner-Mindlin plate model. First by adapting the theory of Chapter 3, we obtain the interior estimate for the Stokes-like system. This is later used to prove that the Arnold-Falk element achieves (almost) first order convergence rate uniformly in the plate thickness t in any interior region. Note that first order convergence cannot be achieved globally (for the soft simply supported plate), due to the existence of a boundary layer in the exact solution. This problem does not arise for the hard clamped boundary conditions considered in [3], since in that case the boundary layer is weaker, and global first order convergence is achieved. Numerical results are given, which confirm the theoretical prediction.

Finally, in the Appendices, we prove two technical results, one about approximation property of linear finite elements and the other about the regularity of the exact solution of the Stokes-like system. They are required in Chapter 4.

CHAPTER 2

INTERIOR ESTIMATES FOR A NONCONFORMING METHOD

2.1 Introduction

In this chapter, we first introduce some standard notations and then take the Poisson equation as an example to study the interior error estimate for the nonconforming finite element method. The linear element will be the focus of the study and the result obtained here will be used in Chapter 4 in the interior estimate of the Arnold-Falk element for the Reissner-Mindlin plate model. We note that more general results can be obtained similarly.

The technique used is a combination of those in [33] and [34]. Even though the method of getting interior estimates for finite element methods is well known (for a comprehensive review, see [41]), the result proven here, to the author's knowledge, is new. We mention that in 1990 Zhan and Wang [44] obtained interior estimates for a class of (compensated) nonconforming elements for second order elliptic equations. However, the method they considered there excludes most standard nonconforming methods, including the linear element we will study here.

Overall, the structure of this chapter is quite similar to that in [33] for the continuous element. So this chapter can serve to review the standard procedure of obtaining interior estimates. Some differences still exist: (1) in section 2.4, additional terms have to be taken into account due to the discontinuity of finite element functions across element edges; (2) in section 2.5, the integration by parts technique, which is essential in Nitsche and Schatz's treatment for the continuous element [33, section 5], is not used. Instead, we use a method by Schatz in [34].

The remainder of this chapter is organized as follows. Section 2.2 presents notations and the model equation. Section 2.3 gives a brief introduction to the nonconforming finite element space and proves some of its properties. Section 2.4 derives an interior duality estimate and section 2.5 shows the final result.

2.2 Notations and Preliminaries

The notations used in this chapter (as well as the whole thesis) are quite standard. For those for Sobolev spaces, cf. Adams [1].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. $L^p(\Omega)$ is the usual space consisting of p-th power integrable functions. $W^{m,p}(\Omega)$ will be the standard Sobolev space of index (m,p) with norm denoted by $\|\cdot\|_{m,p,\Omega}$, for $m \in \mathbb{N}$. The fractional spaces can be defined by interpolation [32]. We shall use the usual L^2-based Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$, with norms denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. Notation $|\cdot|_{s,\Omega}$ denotes the semi-norm of $H^s(\Omega)$. We will drop Ω and use H^s to denote $H^s(\Omega)$, with norm $\|\cdot\|_m$, whenever no confusion can arise. The space $\mathring{H}^s$ is the completion of $C_0^\infty(\Omega)$ in H^s.

For $s \geq 0$, H^{-s} denotes the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{-s,\Omega} = \sup_{\substack{v \in \mathring{H}^s \\ v \neq 0}} \frac{(u,v)}{\|v\|_{s,\Omega}}.$$

The notation $(\cdot\,,\cdot)$ stands for both the L^2 inner product and its extension to a pairing of $\mathring{H}^s$ and H^{-s}. The notation $\langle\cdot\,,\cdot\rangle$ denotes the pairing of $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$. We use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. This is illustrated in the definitions of the following standard differential operators:

$$\operatorname{div} \boldsymbol{\phi} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y}, \quad \mathbf{grad}\, p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}.$$

The letter C denotes a generic constant, not necessarily the same in each occurrence.

Consider the boundary value problem

$$-\Delta u = K - \operatorname{div} \boldsymbol{F} \quad \text{in } \Omega, \tag{2.2.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.2.2}$$

In the above, we include $\operatorname{div} \boldsymbol{F}$ on the right hand side since it appears in a reformulation of the Reissner-Mindlin plate equations for which we will study in Chapter 4. This plate model was the original motivation for the current investigation.

The weak variational form is:

Find $u \in \mathring{H}^1$ such that

$$(\mathbf{grad}\, u, \mathbf{grad}\, v) = (K, v) + (\mathbf{grad}\, v, \boldsymbol{F}) \quad \text{for all } v \in \mathring{H}^1. \tag{2.2.3}$$

From the standard theory on elliptic boundary value problems (cf. [32]), we have:

Lemma 2.2.1. *For a smooth Ω, a given $K \in H^k$, and an $\boldsymbol{F} \in \boldsymbol{H}^{k+1}$, there is a unique solution u satisfying* (2.2.1) *and* (2.2.2). *Moreover,*

$$\|u\|_{k+2} \le C\big(\, \|K\|_k + \|\boldsymbol{F}\|_{k+1} \,\big), \tag{2.2.4}$$

where C is independent of K, $\boldsymbol{F}$, and u.

2.3 The Nonconforming P^1 Element

The notations and definitions for finite element spaces used here follow closely those by Ciarlet [16]. For simplicity, we will assume that Ω is a polygonal domain.

This is just to avoid explaining the construction of curved elements near the boundary $\partial\Omega$. The theory of the interior estimate to be developed in this chapter, however, is independent of this assumption.

By a triangulation of Ω we mean a set $\mathcal{T}_h$ of closed triangles such that the intersection of any two triangles is either a common edge, a common vertex, or empty, and such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. For any $K \in \mathcal{T}_h$, let h_K be its diameter and ρ_K the radius of the largest inscribed disk inside K. Define $h = \max_{K\in\mathcal{T}_h} h_K$.

We will assume that triangulation $\mathcal{T}_h$ is *quasi-uniform* (cf. [31, page 141]), i.e., there are positive constants β_1 and β_2 independent of h such that

$$h_K \geq \beta_1 h, \qquad \frac{\rho_K}{h_K} \geq \beta_2,$$

for all $K \in \mathcal{T}_h$. This restriction carries over to the whole thesis unless otherwise stated.

Define

$$\begin{aligned}
W_h &= \{w \in L^2 : w|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h,\ w \text{ is continuous at midpoints} \\
&\qquad \text{of element edges}\}, \\
\bar{W}_h &= \{w \in L^2 : w|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h,\ w \text{ is continuous at midpoints} \\
&\qquad \text{of element edges and vanishes at midpoints of boundary edges}\}, \\
V_h &= \{v \in H^1 : v|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h,\ v \text{ is continuous at element} \\
&\qquad \text{vertices}\}.
\end{aligned}$$

Here $P_1(T)$ is the set of linear functions on T. The sets W_h and V_h are the standard nonconforming linear finite element space and the conforming linear finite element space, respectively. For $\Omega_0 \subseteq \Omega$, let

$$\mathring{W}_h(\Omega_0) = \{p \in W_h \mid \operatorname{supp} p \subseteq \bar{\Omega}_0\}, \qquad \mathring{V}_h(\Omega_0) = \{v \in V_h \mid \operatorname{supp} v \subseteq \bar{\Omega}_0\}.$$

If $G_h \Subset \Omega$ is a union of triangles, let $W_h(G_h)$, $\bar{W}_h(G_h)$, and $V_h(G_h)$ be defined the same way as W_h, $\bar{W}_h$, and V_h, respectively.

Let G_0 and G be two concentric open disks with $G_0 \Subset G \Subset \Omega$, i.e., $\bar{G}_0 \subset G$ and $\bar{G} \subset \Omega$. Then there is a positive number h_0, such that for $h \leq h_0$, the following properties hold.

Superapproximation property. Let $\omega \in C_0^\infty(G_0)$ and $u \in W_h$. There exists a function $v \in \mathring{W}_h(G)$, such that

$$\| \mathbf{grad}_h(\omega^2 u - v)\|_{0,G} \leq Ch\big(\|\omega \, \mathbf{grad}_h \, u\|_{0,G} + \|u\|_{0,G} \big), \tag{2.3.1}$$

for $C = C(G_0, G, \omega)$. Here for $\mu \in W_h$, $\mathbf{grad}_h \, \mu$ denotes the function with values in the space of piecewise constants that coincides element-wise with $\mathbf{grad} \, \mu$.

Inverse inequality property. Let t be a nonnegative integer. Then there exists a set G_h, which is a union of triangles and satisfies $G_0 \Subset G_h \Subset G$, such that

$$\|u\|_{1,G_h}^h \leq Ch^{-t-1}\|u\|_{-t,G_h} \quad \text{for all } u \in W_h, \tag{2.3.2}$$

where the constant C is independent of h and u. Here $\|u\|_{1,G_h}^h = (\| \mathbf{grad}_h \, u\|_{0,G_h}^2 + \|u\|_{0,G_h}^2)^{1/2}$ for $u \in W_h$.

The above *superapproximation property* is somewhat different from the one in [33] (cf. section 3.3). This is because a different approach will be used in the step of "*interior error estimates*" [33, section 5] (from that for the conforming elements). We mention that (2.3.1) was first proved by Schatz [34] for the continuous linear element and the same proof can be carried over to the nonconforming element. For the sake of completeness, we include the proof.

Proof of the superapproximation property. As $\omega \in C_0^\infty(G_0)$, for h small enough, we can find a set G_h, a union of triangles, such that $G_0 \Subset G_h \Subset G$. By the standard approximation theory on finite element spaces (cf. [16, page 121]), the linear function v_T which interpolates $\omega^2 u$ at the midpoints of the edges of T satisfies

$$\begin{aligned}\|\mathbf{grad}(\omega^2 u - v_T)\|_{0,T} &\le C_T h_T \|\mathbf{grad}(\omega^2 u)\|_{1,T} \\ &\le C_T h_T (\|\omega\, \mathbf{grad}\, u\|_{0,T} + \|u\|_{0,T}),\end{aligned}$$

for each $T \in G_h$. Define v in $W_h(G_h)$ by $v|_T = v_T$ and extend it outside G_h by zero so that $v \in \mathring{W}_h(G)$. Summing up inequalities of above type for each T and using the fact that $\mathcal{T}_h$ is quasi-uniform, we obtain (2.3.1). $\square$

An inverse inequality like (2.3.2) was used in [33] for continuous elements, where it was stated that it could be obtained by using the inverse inequality $\|u\|_{t,G_h} \le Ch^{s-t}\|u\|_{s,G_h}$, for $0 \le s \le t$, and Green's formula. It is unlikely that this approach can be easily adapted for discontinuous elements. In the following we give a proof that is independent of the specific finite element space.

Proof of the inverse inequality. The proof uses a result by Schatz and Wahlbin [35, Lemma 1.1].

Let $t \ge 0$ be an integer. Furthermore, let Ω_j, $j = 1, \ldots, J$, be disjoint open sets with $\overline{\Omega} = \bigcup_{j=1}^{J} \overline{\Omega}_j$. Then

$$\sum_{j=1}^{J} \|u\|^2_{-t,\Omega_j} \le \|u\|^2_{-t,\Omega}, \quad \textit{for all } u \in H^{-t}.$$

Based on the above inequality and the standard inverse inequality [16, page 112]

$$\|u\|_1^h \le Ch^{-1}\|u\|_0 \quad \text{for all } u \in W_h,$$

it is easily seen that one need only prove

$$\|u\|_{0,K} \le Ch^{-t}\|u\|_{-t,K}, \quad \text{for all } u \in P_1(K) \text{ and } K \in \mathcal{T}_h. \tag{2.3.3}$$

To do so, we apply the scaling argument [16]. Let $\hat{K}$ be the standard reference triangle and F_K an affine mapping from $\hat{K}$ into K. For any function $v \in L^2(K)$, let $\hat{v}(\hat{x}) = v(x)$, where $x = F_K(\hat{x})$. Under F_K, the set $W_h(K)$ will be mapped onto $P_1(\hat{K})$, the space of linear polynomials on $\hat{K}$. Using the equivalence of norms on a finite dimensional linear space, we obtain

$$\|\hat{u}\|_{0,\hat{K}} \leq C\|\hat{u}\|_{-t,\hat{K}} \quad \text{for all } \hat{u} \in P_1(\hat{K}), \tag{2.3.4}$$

with C independent of $\hat{u}$. By definition

$$\|\hat{u}\|_{-t,\hat{K}} = \sup_{\substack{\hat{v}\in \mathring{H}^t(\hat{K}) \\ \hat{v}\neq 0}} \frac{(\hat{u},\hat{v})_{\hat{K}}}{\|\hat{v}\|_{t,\hat{K}}}. \tag{2.3.5}$$

We have (cf. [16, page 140])

$$(\hat{u},\hat{v})_{\hat{K}} \leq Ch_K^{-2}(u,v)_K \tag{2.3.6}$$

and

$$\|\hat{v}\|_{t,\hat{K}} \geq Ch_K^{-1}\Big(\sum_{i=0}^{t} h_K^{2i}|v|_{i,K}^2\Big)^{\frac{1}{2}} \geq Ch_K^{t-1}\|v\|_{t,K}, \tag{2.3.7}$$

with constant C depending only on the minimum angle of K. Substituting (2.3.6) and (2.3.7) into (2.3.5) yields

$$\|\hat{u}\|_{-t,\hat{K}} \leq Ch_K^{-t-1} \sup_{\substack{v\in \mathring{H}^t(K) \\ v\neq 0}} \frac{(u,v)}{\|v\|_{t,K}} = Ch_K^{-t-1}\|u\|_{-t,K}.$$

Since

$$\|u\|_{0,K} \leq Ch_K\|\hat{u}\|_{0,\hat{K}}$$

and the mesh is quasi-uniform, inequality (2.3.3) follows. $\square$

The finite element approximation for (2.2.1) and (2.2.2) is:

Find a $u_h \in \bar{W}_h$ such that

$$(\mathbf{grad}_h\, u_h, \mathbf{grad}_h\, v) = (K, v) + (\mathbf{grad}_h\, v, \boldsymbol{F}) \quad \text{for all } v \in \bar{W}_h. \tag{2.3.8}$$

The following convergence theorem is well known. See, for example [3, Lemma 5.4].

Lemma 2.3.1. *Let $K \in L^2$ and $\boldsymbol{F} \in \boldsymbol{H}^1$. Assume that Ω is a convex polygon and u and u_h are the solutions of* (2.2.1) *and* (2.2.2)*, and* (2.3.8)*, respectively. Then,*

$$\|u - u_h\|_0 + h\|\,\mathbf{grad}_h(u - u_h)\|_0 \leq Ch^2\big(\,\|K\|_0 + \|\boldsymbol{F}\|_1\,\big). \tag{2.3.9}$$

The following estimate, which can be found in [18], will play an important role in our analysis.

Lemma 2.3.2. *There is a constant C independent of h such that*

$$\left|\sum_{T \in \mathcal{T}_h} \int_{\partial T} u\boldsymbol{w} \cdot \boldsymbol{n}_T\right| \leq Ch\|\boldsymbol{w}\|_1 \inf_{v \in \mathring{H}^1} \|\,\mathbf{grad}_h(u - v)\|_0$$

$$\text{for all } \boldsymbol{w} \in \boldsymbol{H}^1,\ u \in \bar{W}_h + \mathring{H}^1, \tag{2.3.10}$$

where $\boldsymbol{n}_T$ is the outer normal of each triangle T.

Before we turn to the next section, we define a semi-norm for linear functional L on $\mathring{W}_h(G)$.

$$\|L\|_G = \sup_{\substack{v \in \mathring{W}_h(G) \\ \mathbf{grad}_h\, v \neq 0}} \frac{L(v)}{\|\,\mathbf{grad}_h\, v\|_{0,G}}.$$

We also want to point out that the results of this chapter require that the mesh size h to be sufficiently small (which is self-evident from the analysis involved). However, for the sake of simplicity, we may not mention it explicitly.

2.4 An Interior Duality Estimate

In this section we will derive an interior duality estimate. The method used here is parallel to that for the conforming method, but there are some additional terms, which measure the jumps of the discontinuous finite elements, to be taken care of.

Let $u \in H^1$ be some solution to the Poisson equation (2.2.1) and $u_h \in W_h$ be some finite element solution satisfying (both without regard to the boundary conditions)

$$(\mathbf{grad}_h\, u_h, \mathbf{grad}_h\, v) = (K, v) - (\mathbf{grad}_h\, v, \boldsymbol{F}) \quad \text{for all } v \in \mathring{W}_h.$$

Using integration by parts we obtain

$$(\mathbf{grad}_h(u - u_h), \mathbf{grad}_h\, v) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\frac{\partial u}{\partial n} - \boldsymbol{F} \cdot \boldsymbol{n}_T)v \quad \text{for all } v \in \mathring{W}_h. \tag{2.4.1}$$

The interior error analysis only depends on the above interior discretization equation.

Lemma 2.4.1. *Let L be a linear functional on W_h and assume that $u \in H^1 + W_h$ satisfies*

$$\big(\, \mathbf{grad}_h\, u, \mathbf{grad}_h\, v\, \big) = L(v) \quad \textit{for all } v \in \mathring{V}_h. \tag{2.4.2}$$

Then for any concentric disks $G_0 \Subset G \Subset \Omega$ and any nonnegative integer t

$$\|u\|_{0,G_0} \le C\big(\, h\|\, \mathbf{grad}_h\, u\|_{0,G} + \|u\|_{-t,G} + \|L\|_G\, \big). \tag{2.4.3}$$

Moreover, if $L(v) = 0$ for all $v \in \mathring{V}_h$, then

$$\|u\|_{0,G_0} \le C\big(\, h\|\, \mathbf{grad}_h\, u\|_{0,G} + \|u\|_{-t,G}\, \big). \tag{2.4.4}$$

Proof. We first prove that for any integer $s \ge 0$,

$$\|u\|_{-s,G_0} \le C\big(\, h\|\, \mathbf{grad}_h\, u\|_{0,G} + \|u\|_{-s-1,G} + \|L\|_G\, \big) \tag{2.4.5}$$

holds for any concentric disks $G_0 \Subset G \Subset \Omega$ (not necessarily the same sets as in (2.4.3)). Then inequality (2.4.3) can be obtained by iteration.

Find a union of elements G_h, such that $G_0 \Subset G_h \Subset G$. Construct a cut-off function $\omega \in C_0^\infty(G_h)$ such that $\omega = 1$ on G_0. By definition

$$\|u\|_{-s,G_0} \le \|\omega u\|_{-s,G} = \sup_{\substack{\phi \in \mathring{H}^s(G)\\ \phi \neq 0}} \frac{(\omega u, \phi)}{\|\phi\|_{s,G}}. \tag{2.4.6}$$

By Lemma 2.2.1, there exists a unique function $U \in H^{s+2}(G) \cap \mathring{H}^1(G)$, such that

$$-\Delta U = \phi \quad \text{in } G,$$

$$U = 0 \quad \text{on } \partial G.$$

Moreover,

$$\|U\|_{s+2,G} \le C\|\phi\|_{s,G}. \tag{2.4.7}$$

For convenience, we extend U by zero outside the disk G. Now, we can estimate the numerator of the right hand side of (2.4.6):

$$\begin{aligned}
(\omega u, \phi)_G &= -(\omega u, \Delta U)_{G_h} \\
&= \sum_{T \in G_h} \left[(\mathbf{grad}(\omega u), \mathbf{grad}\, U)_T - \int_{\partial T} \omega u \frac{\partial U}{\partial n} ds \right] \\
&= \sum_{T \in G_h} \Big[(\mathbf{grad}\, u, \mathbf{grad}(\omega U))_T + (u\, \mathbf{grad}\, \omega, \mathbf{grad}\, U)_T \\
&\qquad + (u, \operatorname{div}(U\, \mathbf{grad}\, \omega))_T - \int_{\partial T} (uU \frac{\partial \omega}{\partial n} + \omega u \frac{\partial U}{\partial n}) ds \Big],
\end{aligned} \tag{2.4.8}$$

where we use the definition of U, differentiation rules, and integration by parts. Since $\operatorname{supp} \omega U \subset G_h \subset G$, the continuous piecewise linear interpolant $(\omega U)^I$ of ωU belongs to $\in \mathring{V}_h(G)$, thus

$$\|\omega U - (\omega U)^I\|_{1,G} \le Ch\|U\|_{2,G}. \tag{2.4.9}$$

So we have

$$
\begin{aligned}
(\omega u, \phi) = & \sum_{T\in G_h} \left[\big(\mathbf{grad}\, u, \mathbf{grad}(\omega U - (\omega U)^I \,\big)_T \right] + L\big((\omega U)^I \,\big) \\
& + \sum_{T\in G_h} \left[\big(\, u\, \mathbf{grad}\, \omega, \mathbf{grad}\, U \,\big)_T + \big(\, u, \operatorname{div}(U\, \mathbf{grad}\, \omega) \,\big)_T \right] \\
& - \sum_{T\in G_h} \left[\int_{\partial T} uU \frac{\partial \omega}{\partial n} ds + \int_{\partial T} \omega u \frac{\partial U}{\partial n} ds \right] \\
=: & \; A + L\big((\omega U)^I \,\big) + B + C. \qquad (2.4.10)
\end{aligned}
$$

where we use the fact that $\big(\mathbf{grad}_h\, u, \mathbf{grad}(\omega U)^I \,\big) = L\big((\omega U)^I \,\big)$. Applying (2.4.9), Lemma 2.3.2, and the Schwarz inequality, we get

$$
\begin{aligned}
|A| &\le Ch \| \mathbf{grad}_h\, u \|_{0,G_h} \|U\|_{2,G}, \\
|B| &\le C \|u\|_{-s-1,G} \|U\|_{s+2,G}, \\
|C| &\le Ch \| \mathbf{grad}_h\, u \|_{0,G_h} \|U\|_{2,G}, \\
|L\big((\omega U)^I \,\big)| &\le C \|L\|_{G_h} \|U\|_{1,G}.
\end{aligned} \qquad (2.4.11)
$$

Substituting (2.4.11) into (2.4.10), then using (2.4.6) and (2.4.7) we obtain (2.4.5).

To prove (2.4.3), take a family of concentric disks: $G_0 \Subset G_1 \Subset \dots G_t = G$. Then applying (2.4.5) with $s = 0$ and G replaced by G_1, we obtain

$$
\|u\|_{0,G_0} \le C\big(h \| \mathbf{grad}_h\, u \|_{0,G_1} + \|u\|_{-1,G_1} + \|L\|_{G_1} \big).
$$

To bound $\|u\|_{-1,G_1}$, we apply (2.4.5) with G_0 and G replaced by G_1 and G_2, respectively, and $s = 1$. Thus, we get

$$
\|u\|_{0,G_0} \le C\big(h \| \mathbf{grad}_h\, u \|_{0,G_2} + \|u\|_{-2,G_2} + \|L\|_{G_2} \big).
$$

Continuing in this fashion, we obtain the (2.4.3).

If $L(v) = 0$ for all $v \in \mathring{V}_h$, we see easily from the above proof that the term $\|L\|_G$ can be taken away from the right hand side of (2.4.3). $\square$

2.5 The Main Result

In this section, we prove the main result of this chapter: Theorem 2.5.2. To be specific, we first use a local energy estimate to study the discrete function satisfying (2.5.1). This equation is usually satisfied by the difference between the global finite element solution and the local projection of the exact solution. Then we combine it with Lemma 2.4.1 to obtain a local version of Theorem 2.5.2. The final result is obtained by a covering argument.

Lemma 2.5.1. *Let L be a linear functional on W_h and assume that $u \in W_h$ satisfies*

$$(\mathbf{grad}_h\, u, \mathbf{grad}_h\, v) = L(v) \quad \text{for all } v \in \mathring{W}_h. \tag{2.5.1}$$

Then for any concentric disk $G_0 \Subset G$ and nonnegative integer t, the following holds

$$\|u\|^h_{1,G_0} \le C\big(\, \|u\|_{-t,G} + \|L\|_G \,\big). \tag{2.5.2}$$

Proof. Let $G_0 \Subset G_1 \Subset G$ be concentric disks and G_h, $G_1 \Subset G_h \Subset G$, be a union of triangles. Construct a cut-off function $\omega \in C_0^\infty(G_1)$ such that $\omega = 1$ on G_0. Then,

$$\begin{aligned} \|\,\mathbf{grad}_h\, u\|^2_{0,G_0} &\le \|\omega\, \mathbf{grad}_h\, u\|^2_{0,G} = \int_G \omega^2\, \mathbf{grad}_h\, u \cdot \mathbf{grad}_h\, u \\ &= \left\{ \int_G \mathbf{grad}_h\, u \cdot \mathbf{grad}_h (\omega^2 u) \right\} - \left\{ 2 \int_G \omega\, \mathbf{grad}_h\, u \cdot u\, \mathbf{grad}\, \omega \right\} = J_1 + J_2. \end{aligned} \tag{2.5.3}$$

Using the inverse inequality (cf. [16])

$$h\|\,\mathbf{grad}_h\, u\|_{0,G_h} \le C\|u\|_{0,G_h}, \tag{2.5.4}$$

the Schwarz inequality, (2.3.1), and the arithmetic-geometric mean inequality, we get

$$
\begin{aligned}
\mid J_1 \mid &= \mid \int_G \mathbf{grad}_h\, u \cdot \mathbf{grad}_h \big(\omega^2 u - (\omega^2 u)^I \big) + L\big((\omega^2 u)^I \big) \mid \\
&\le C \| \mathbf{grad}_h\, u \|_{0,G_h} \| \mathbf{grad}_h \big(\omega^2 u - (\omega^2 u)^I \big) \|_{G_h} + \|L\|_{G_1} \| \mathbf{grad}_h (\omega^2 u)^I \|_{0,G_1} \\
&\le C h \| \mathbf{grad}_h\, u \|_{0,G_h} \big(\|\omega\, \mathbf{grad}_h\, u\|_{0,G_1} + \|u\|_{0,G_1} \big) \\
&\quad + \|L\|_{G_1} \big(\| \mathbf{grad}_h (\omega^2 u)\|_{0,G_1} + \| \mathbf{grad}_h \big(\omega^2 u - (\omega^2 u)^I \big) \|_{0,G_1} \big) \\
&\le \frac{1}{4} \|\omega\, \mathbf{grad}_h\, u\|_{0,G}^2 + C\|u\|_{0,G_1}^2 + C\|L\|_G^2. \qquad (2.5.5)
\end{aligned}
$$

The estimate on $|J_2|$ is straightforward:

$$
\begin{aligned}
\mid J_2 \mid &\le C \|\omega\, \mathbf{grad}_h\, u\|_{0,G_1} \|u\|_{0,G_1} \\
&\le \frac{1}{4} \|\omega\, \mathbf{grad}_h\, u\|_{0,G}^2 + C\|u\|_{0,G_1}^2.
\end{aligned} \qquad (2.5.6)
$$

Combining (2.5.3), (2.5.5), and (2.5.6), then taking the square root, we obtain

$$
\| \mathbf{grad}_h\, u \|_{0,G_0} \le C \big(\|u\|_{0,G_1} + \|L\|_G \big).
$$

From (2.5.1) and Lemma 2.4.1, we have

$$
\|u\|_{0,G_0} \le C (h \| \mathbf{grad}_h\, u\|_{0,G} + \|u\|_{-t,G} + \|L\|_G).
$$

Combing the above two inequalities we get

$$
\|u\|_{1,G_0}^h \le C \big(\|u\|_{0,G_1} + h\|u\|_{1,G}^h + \|u\|_{-t,G} + \|L\|_G \big).
$$

Then using Lemma 2.4.1 again with G_0 replaced by G_1 to bound $\|u\|_{0,G_1}$ on the right hand side of the above inequality yields

$$
\|u\|_{1,G_0}^h \le C \big(h\|u\|_{1,G}^h + \|u\|_{-t,G} + \|L\|_G \big). \qquad (2.5.7)
$$

We will now use an iteration method [33] to prove (2.5.2).

Let $G_0 \Subset G_1 \Subset \dots \Subset G_{t+2} = G$ be concentric disks and apply (2.5.7) to each pair $G_j \Subset G_{j+1}$ (with G_0 and G replaced by G_j and G_{j+1}, respectively) to get

$$\|u\|^1_{1,G_j} \le C\big(\, h\|u\|^h_{1,G_{j+1}} + \|u\|_{-t,G_{j+1}} + \|L\|_{G_{j+1}}\,\big).$$

Combining inequalities of above type (for $j = 0,\ 1, \dots$) we obtain

$$\begin{aligned}\|u\|^h_{1,G_0} &\le C\big(\, h\|u\|^h_{1,G_1} + \|u\|_{-t,G_1} + \|L\|_{G_1}\,\big)\\ &\le \dots\\ &\le C\big(\, h^{t+1}\|u\|^h_{1,G_{t+1}} + \|u\|_{-t,G_{t+1}} + \|L\|_{G_{t+1}}\,\big).\end{aligned}$$

By (2.3.2), there is a set G_h, $G_{t+1} \Subset G_h \Subset G_{t+2} = G$, such that

$$h^{t+1}\|u\|^h_{1,G_{t+1}} \le h^{t+1}\|u\|^h_{1,G_h} \le C\|u\|_{-t,G_h} \le C\|u\|_{-t,G}.$$

Thus the above two inequalities imply (2.5.2). $\square$

Theorem 2.5.2. *Let $\Omega_0 \Subset \Omega_1 \Subset \Omega$ and assume that $K \in L^2$ and $\operatorname{div} \boldsymbol{F} \in L^2$. Assume that $\boldsymbol{F}|_{\Omega_1} \in \boldsymbol{H}^1(\Omega_1)$. Suppose that $u \in H^1$ satisfies $u|_{\Omega_1} \in H^2(\Omega_1)$ and $u_h \in W_h$ satisfies* (2.4.1). *Let t be a nonnegative integer. Then there exists a constant C depending only on Ω_0, Ω_0, and t, and a positive number h_1, such that for $h \in (0, h_1]$*

$$\|u - u_h\|^h_{1,\Omega_0} \le C\big(\, h\|u\|_{2,\Omega_1} + h\|\boldsymbol{F}\|_{1,\Omega_1} + \|u - u_h\|_{-t,\Omega_1}\,\big), \qquad (2.5.8)$$

$$\|u - u_h\|_{0,\Omega_0} \le C\big(\, h^2\|u\|_{2,\Omega_1} + h^2\|\boldsymbol{F}\|_{1,\Omega_1} + \|u - u_h\|_{-t,\Omega_1}\,\big). \qquad (2.5.9)$$

Proof. We first prove a local version of (2.5.8), that is,

$$\|u - u_h\|^h_{1,G_0} \le C\, h\big(\, \|u\|_{2,G} + \|\boldsymbol{F}\|_{1,G}\,\big) + C\|u - u_h\|_{-t,G}, \qquad (2.5.10)$$

for any pair of concentric disks $G_0 \Subset G \Subset \Omega$. In order to do so, we find a disk G_1 such that $G_0 \Subset G_1 \Subset G_h \Subset G$, with G_h a union of triangles. Construct a cut-off function $\omega \in C_0^\infty(G_h)$ such that $\omega = 1$ on G_1. Use the notation $\widetilde{u} = \omega u$ and define $\pi\widetilde{u} \in \bar{W}_h(G_h)$ by

$$(\mathbf{grad}_h\, \pi\widetilde{u}, \mathbf{grad}_h\, v) = (\mathbf{grad}_h\, \widetilde{u}, \mathbf{grad}_h\, v) \quad \text{for all } v \in \bar{W}_h(G_h). \tag{2.5.11}$$

This problem is uniquely solvable. Moreover,

$$\|\widetilde{u} - \pi\widetilde{u}\|_{1,G_h}^h \le C \inf_{v \in \bar{W}_h(G_h)} \|\widetilde{u} - v\|_{1,G_h} \le C\, h \|u\|_{2,G_h}.$$

By the triangle inequality,

$$\begin{aligned} \|u - u_h\|_{1,G_0}^h &\le \|\widetilde{u} - \pi\widetilde{u}\|_{1,G_h}^h + \|\pi\widetilde{u} - u_h\|_{1,G_0}^h \\ &\le C\, h \|u\|_{2,G_h} + \|\pi\widetilde{u} - u_h\|_{1,G_0}^h. \end{aligned} \tag{2.5.12}$$

From (2.5.11), (2.4.1), and the fact that $\omega = 1$ on G_1,

$$\begin{aligned} (\mathbf{grad}_h(\pi\widetilde{u} - u_h), \mathbf{grad}_h\, v) &= (\mathbf{grad}_h(u - u_h), \mathbf{grad}_h\, v) \\ &= \sum_{T \in G_h} \int_{\partial T} \Big(\frac{\partial u}{\partial n} - \boldsymbol{F} \cdot \boldsymbol{n}_T\Big) v \\ &=: L(v) \quad \text{for all } v \in \mathring{W}_h(G_1). \end{aligned}$$

By Lemma 2.3.2,

$$|L(v)_{G_h}| \le C h (\|u\|_{2,G_h} + \|\boldsymbol{F}\|_{1,G_h}) \|\,\mathbf{grad}_h\, v\|_{0,G_h} \quad \text{for all } v \in \mathring{W}_h(G_h),$$

which implies

$$\|L\|_{G_h} \le C\, h\big(\, \|u\|_{2,G_h} + \|\boldsymbol{F}\|_{1,G_h}\,\big). \tag{2.5.13}$$

Then, applying Lemma 2.5.1 with G replaced by G_1, we obtain

$$\begin{aligned}
\|\pi\widetilde{u} - u_h\|^h_{1,G_0} &\le C\big(\, \|\pi\widetilde{u} - u_h\|_{-t,G_1} + \|L\|_{G_1}\big) \\
&\le C(\|\pi\widetilde{u} - \widetilde{u}\|_{-t,G_h} + \|u - u_h\|_{-t,G_1} + \|L\|_{G_h}) \\
&\le C\,h(\|\boldsymbol{F}\|_{1,G} + \|u\|_{2,G}\,) + C\|u - u_h\|_{-t,G}. \qquad (2.5.14)
\end{aligned}$$

By the triangle inequality, (2.5.12), and (2.5.14), inequality (2.5.10) is obtained. In order to prove (2.5.8), we use a covering argument. Let $d = d_0/2$ where $d_0 = \mathrm{dist}(\,\bar{\Omega}_0, \partial\Omega_1)$. Cover $\bar{\Omega}_0$ with a finite number of disks $G_0(x_i)\,,\ i = 1, 2, \ldots, m$ centered at $x_i \in \bar{\Omega}_0$ with $\mathrm{diam}\, G_0(x_i) = d$. Let $G(x_i)\,,\ i = 1, 2, \ldots, k$ be corresponding concentric disks with $\mathrm{diam}\, G(x_i) = 2d$. Applying (2.5.10) to each pair $G_0(x_i)$ and $G(x_i)$, and adding inequalities of the form (2.5.10), we obtain the desired result.

To prove (2.5.9), note that

$$\big(\,\mathbf{grad}_h(\pi\widetilde{u} - u_h), \mathbf{grad}\, v\,\big) = 0 \quad \text{for all } v \in \mathring{V}_h(G_1).$$

By Lemma 2.4.1, we obtain

$$\|u - u_h\|_{0,G_0} \le C\big(\,h\|\,\mathbf{grad}_h(u - u_h)\|_{0,G_1} + \|u - u_h\|_{-t,G_1}\,\big),$$

for any disks $G_0 \Subset G_1$. Then, applying (2.5.10) with G_0 replaced by G_1 to get

$$\|u - u_h\|_{0,G_0} \le C\big(\,h^2\|u\|_{2,G} + h^2\|\boldsymbol{F}\|_{1,G} + \|u - u_h\|_{-t,G}\,\big),$$

for any pair of disks $G_0 \Subset G \Subset \Omega$. Then a covering argument leads to (2.5.9) $\square$

CHAPTER 3

INTERIOR ESTIMATES FOR THE STOKES EQUATIONS

3.1 Introduction

In this chapter we establish interior error estimates for finite element approximations to solutions of the Stokes equations. The theory (cf. [6]) to be developed here covers a wide range of finite element methods for the Stokes equations. It is based on some abstract hypotheses that apply to most stable elements. This is different than what we did in Chapter 2, where we only studied one special element.

The conclusion we obtain here is quite similar to that for the second order elliptic equation. Namely, we prove that, the approximation error of the finite element method in the interior region is bounded above by two terms: the first one measures the local approximability of the exact solution by the finite element space and the second one, given in an arbitrary weak Sobolev norm over a slightly larger domain, represents a global pollution effect.

The technique used here is adapted from that for the second order elliptic equation by Nitche and Schatz [33]. Although the general approach is not new, there are a number of significant difficulties which arise for the Stokes system that are not present in previous works. The method developed here will also be generalized to get the interior error estimate of the Arnold-Falk element for the Reissner-Mindlin plate model in the next chapter.

After the preliminaries of the next section, we set out the hypotheses for the finite element spaces in section 3.3. These assumptions are satisfied by most stable elements on a locally quasi-uniform mesh. In section 3.4, we introduce the local equations and derive some basic properties of their solutions. Section 3.5 gives the precise statement of our main result and its proof. In section 3.6, we apply the general theory to the MINI element of Arnold-Brezzi-Fortin [2] and show that it achieves the optimal convergence rate in the energy norm away from the boundary for a nonconvex polygonal domain. However this optimal convergence cannot be obtained on the whole domain due to the corner singularity of the exact solution.

3.2 Notations and Preliminaries

Let Ω denote a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We define the gradient of a vector function:

$$\mathbf{grad}\,\boldsymbol{\phi} = \begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_1/\partial y \\ \partial\phi_2/\partial x & \partial\phi_2/\partial y \end{pmatrix}.$$

Let G be an open subset of Ω and s an integer. If $\boldsymbol{\phi} \in H^s(G)$, $\boldsymbol{\psi} \in H^{-s}(G)$, and $\omega \in C_0^\infty(G)$, then

$$|(\omega\boldsymbol{\phi}, \boldsymbol{\psi})| \le C\|\boldsymbol{\phi}\|_{s,G}\|\boldsymbol{\psi}\|_{-s,G},$$

with the constant C depending only on G, ω, and s. For $\boldsymbol{\Phi} \in \boldsymbol{H}^s(G)$, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-s+1}(G)$ define

$$\mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\Psi}) = \big(\, \boldsymbol{\Phi}(\mathbf{grad}\,\omega)^t, \mathbf{grad}\,\boldsymbol{\Psi}\,\big) - \big(\,\mathbf{grad}\,\boldsymbol{\Phi}, \boldsymbol{\Psi}(\mathbf{grad}\,\omega)^t\,\big). \tag{3.2.1}$$

Then

$$|\mathrm{R}\big(\,\omega, \boldsymbol{\Phi}, \boldsymbol{\Psi}\,\big)| \le C\|\boldsymbol{\Phi}\|_{s,G}\|\boldsymbol{\Psi}\|_{-s+1,G}. \tag{3.2.2}$$

If, moreover, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-s+2}$, we have the identity

$$\big(\,\mathbf{grad}(\omega\boldsymbol{\Phi}), \mathbf{grad}\,\boldsymbol{\Psi}\,\big) = \big(\,\mathbf{grad}\,\boldsymbol{\Phi}, \mathbf{grad}(\omega\boldsymbol{\Psi})\,\big) + \mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\Psi}).$$

If X is any subspace of L^2, then $\hat{X}$ denotes the subspace of elements with average value zero.

The following lemma states the well-posedness and regularity of the Dirichlet problem for the generalized Stokes equations on smooth domains. (Because we are interested in local estimates we really only need this results when the domain is a disk.) For the proof see [40, Chapter I, § 2].

Lemma 3.2.1. *Let G be a smoothly bounded plane domain and m a nonnegative integer. Then for any given functions $\boldsymbol{F} \in \boldsymbol{H}^{m-1}(G)$, $K \in H^m(G) \cap \hat{L}^2(G)$, there exist uniquely determined functions*

$$\boldsymbol{\phi} \in \boldsymbol{H}^{m+1}(G) \cap \mathring{\boldsymbol{H}}^1(G), \; p \in H^m(G) \cap \hat{L}^2(G),$$

such that

$$\begin{aligned}\big(\mathbf{grad}\,\boldsymbol{\phi}, \mathbf{grad}\,\boldsymbol{\psi}\big) - \big(\operatorname{div}\boldsymbol{\psi}, p\big) &= \big(\boldsymbol{F}, \boldsymbol{\psi}\big) \quad \textit{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(G),\\ \big(\operatorname{div}\boldsymbol{\phi}, q\big) &= \big(K, q\big) \quad \textit{for all } \; q \in \hat{L}^2(G).\end{aligned}$$

Moreover,

$$\|\boldsymbol{\phi}\|_{m+1,G} + \|p\|_{m,G} \le C\big(\|\boldsymbol{F}\|_{m-1,G} + \|K\|_{m,G}\big),$$

where the constant C is independent of $\boldsymbol{F}$ and K.

3.3 Finite Element Spaces

In this section we collect assumptions on the mixed finite element spaces. As usual for the interior estimate, we require the superapproximation property of the finite element spaces, in addition to the the approximation and stability properties.

Let $\Omega \subset \mathbb{R}^2$ be the bounded open set on which we solve the Stokes equations. We denote by $\boldsymbol{V}_h$ the finite element subspace of $\boldsymbol{H}^1$, and by W_h the finite element subspace of L^2. For $\Omega_0 \subseteq \Omega$, define

$$\boldsymbol{V}_h(\Omega_0) = \{\, \boldsymbol{\phi}|_{\Omega_0} \mid \boldsymbol{\phi} \in \boldsymbol{V}_h \,\}, \qquad W_h(\Omega_0) = \{\, p|_{\Omega_0} \mid p \in W_h \,\},$$
$$\mathring{\boldsymbol{V}}_h(\Omega_0) = \{\, \boldsymbol{\phi} \in \boldsymbol{V}_h \mid \operatorname{supp} \boldsymbol{\phi} \subseteq \bar{\Omega}_0 \}, \qquad \mathring{W}_h(\Omega_0) = \{\, p \in W_h \mid \operatorname{supp} p \subseteq \bar{\Omega}_0 \,\}.$$

Let G_0 and G be concentric open disks with $G_0 \Subset G \Subset \Omega$. We assume that there exists a positive real number h_0 and positive integers k_1 and k_2, such that for $h \in \big(0, h_0\big]$, the following properties hold.

A1. *Approximation property.*

(1) If $\boldsymbol{\phi} \in \boldsymbol{H}^m(G)$ for some positive integer m, then there exists a $\boldsymbol{\phi}^I \in \boldsymbol{V}_h$ such that

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}^I\|_{1,G} \leq C h^{r_1 - 1} \mid \boldsymbol{\phi} \mid_{m,G}, \quad r_1 = \min(k_1 + 1, m).$$

(2) If $p \in H^l(G)$ for some nonnegative integer l, then there exists a $p^I \in W_h$, such that

$$\|p - p^I\|_{0,G} \leq C h^{r_2} \|p\|_{l,G}, \quad r_2 = \min(k_2 + 1, l).$$

Furthermore, if $\boldsymbol{\phi}$ and p vanish on $G \setminus \bar{G}_0$, respectively, then $\boldsymbol{\phi}^I$ and p^I can be chosen to vanish on $\Omega \setminus \bar{G}$.

A2. *Superapproximation property.* Let $\omega \in C_0^\infty(G)$, $\boldsymbol{\phi} \in \boldsymbol{V}_h$, and $p \in W_h$. Then there exist $\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G)$ and $q \in \mathring{W}_h(G)$, such that

$$\|\omega\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,\Omega} \leq Ch\|\boldsymbol{\phi}\|_{1,G},$$
$$\|\omega p - q\|_{0,\Omega} \leq Ch\|p\|_{0,G},$$

where C depends only on G and ω.

A3. *Inverse property.* For each $h \in (0, h_0]$, there exists a set G_h, $G_0 \Subset G_h \Subset G$, such that for each nonnegative integer m there is a constant C for which

$$\|\boldsymbol{\phi}\|_{1,G_h} \le Ch^{-1-m}\|\boldsymbol{\phi}\|_{-m,G_h} \quad \text{for all } \boldsymbol{\phi} \in \boldsymbol{V}_h,$$

$$\|p\|_{0,G_h} \le Ch^{-m}\|p\|_{-m,G_h} \quad \text{for all } p \in W_h.$$

A4. *Stability property.* There is a positive constant γ, such that for all $h \in (0, h_0]$ there is a domain G_h, $G_0 \Subset G_h \Subset G$ for which

$$\inf_{\substack{p \in \hat{W}_h(G_h) \\ p \neq 0}} \sup_{\substack{\boldsymbol{\phi} \in \mathring{\boldsymbol{V}}_h(G_h) \\ \boldsymbol{\phi} \neq 0}} \frac{(\operatorname{div} \boldsymbol{\phi}, p)_{G_h}}{\|\boldsymbol{\phi}\|_{1,G_h}\|p\|_{0,G_h}} \ge \gamma$$

When $G_h = \Omega$, property A4 is the standard stability condition for Stokes elements. It will usually hold as long as G_h is chosen to be a union of elements. The standard stability theory for mixed methods then gives us the following result.

Lemma 3.3.1. *Let G_h be a subdomain for which the stability inequality in A4 holds. Then for $\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(G_h)$ and $p \in L^2(G_h)$, there exist unique $\pi\boldsymbol{\phi} \in \mathring{\boldsymbol{V}}_h(G_h)$ and $\pi p \in W_h(G_h)$ with $\int_{G_h} \pi p = \int_{G_h} p$ such that*

$$\big(\mathbf{grad}(\boldsymbol{\phi} - \pi\boldsymbol{\phi}), \mathbf{grad}\ \boldsymbol{\psi}\big) - \big(\operatorname{div} \boldsymbol{\psi}, p - \pi p\big) = 0 \quad \textit{for all } \ \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h),$$

$$\big(\operatorname{div}(\boldsymbol{\phi} - \pi\boldsymbol{\phi}), q\big) = 0 \quad \textit{for all } q \in W_h(G_h).$$

Moreover,

$$\|\boldsymbol{\phi} - \pi\boldsymbol{\phi}\|_{1,G_h} + \|p - \pi p\|_{0,G_h} \le C\big(\inf_{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h)} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,G_h} + \inf_{q \in W^h(G_h)} \|p - q\|_{0,G_h}\big).$$

The approximation properties A1 are typical of finite element spaces $\boldsymbol{V}_h$ and W_h constructed from polynomials of degrees at least k_1 and k_2, respectively. (It does not matter that the subdomain G is not a union of elements since $\boldsymbol{\phi}$ and p

can be extended beyond G.) The inverse inequality was proved in section 2.3 for general finite element spaces. The superapproximation property is discussed as Assumptions 7.1 and 9.1 in [41]. Many finite element spaces are known to have the superapproximation property. In particular, it was verified in [33] for Lagrange and Hermite elements. To end this section we shall verify the superapproximation for the MINI element.

Let b_T denote the cubic bubble on the triangle T, so on T, b_T is the cubic polynomial satisfying $b_T|_{\partial T} = 0$ and $\int_T b_T = 1$. We extend b_T outside T by zero. For a given triangulation $\mathcal{T}_h$ let V_h denote the span of the continuous piecewise linear functions and the bubble functions b_T, $T \in \mathcal{T}_h$. The MINI element uses $V_h \times V_h$ as the finite element space for velocities. We wish to show that if $\phi \in V_h$ and $\omega \in C_0^\infty(G)$ then $\|\omega\phi - \psi\|_{1,G} \le Ch\|\phi\|_{1,G}$ for some $\psi \in \mathring{V}_h(G)$. We begin by writing $\phi = \phi_l + \phi_b$ with ϕ_l piecewise linear and $\phi_b = \sum_{T\in\mathcal{T}_h} \beta_T b_T$ for some $\beta_T \in \mathbb{R}$.

We know that there exists a piecewise linear function ψ_l supported in G for which

$$\|\omega\phi_l - \psi_l\|_{1,\Omega} \le Ch\|\phi_l\|_{1,G}.$$

Turning to the bubble function term ϕ_b define $\psi_b = \sum_{T\subset G}(\beta_T \mathcal{L}_{\mathrm{T}}\,\omega)b_T \in \mathring{V}_h(G)$ where $\mathcal{L}_{\mathrm{T}}\,\omega \in \mathbb{R}$ is the average value of ω on T. Now if T intersects $\operatorname{supp}\omega$ then $T \subset G$, at least for h sufficiently small. Hence

$$\begin{aligned} \|\omega\phi_b - \psi_b\|_{0,\Omega}^2 &= \sum_{T\subset G} \|\omega\phi_b - \psi_b\|_{0,T}^2 = \sum_{T\subset G} \|\beta_T b_T(\omega - \mathcal{L}_{\mathrm{T}}\,\omega)\|_{0,T}^2 \\ &\le \sum_{T\subset G} \|\omega - \mathcal{L}_{\mathrm{T}}\,\omega\|_{L^\infty(T)}^2 \|\beta_T b_T\|_{0,T}^2 \le Ch^2\|\phi_b\|_{0,G}^2, \end{aligned}$$

where the constant C depends on ω. Moreover,

$$\|\,\mathbf{grad}(\omega\phi_b - \psi_b)\|_{0,\Omega}^2 = \sum_{T\subset G} \|\,\mathbf{grad}(\omega\phi_b - \psi_b)\|_{0,T}^2$$

$$= \sum_{T\subset G} \|\,\mathbf{grad}\big(\,\beta_T b_T(\omega - \mathcal{L}_\mathrm{T}\,\omega)\,\big)\|_{0,T}^2$$

$$= \sum_{T\subset G} \|\beta_T(w - \mathcal{L}_\mathrm{T}\,\omega)\,\mathbf{grad}\,b_T + \beta_T b_T\,\mathbf{grad}(\omega - \mathcal{L}_\mathrm{T}\,\omega)\|_{0,T}^2$$

$$\le C\big(\,h^2 \sum_{T\subset G} \|\,\mathbf{grad}\,\omega\|_{\infty,T}^2 \|\beta_T\,\mathbf{grad}\,b_T\|_{0,T}^2 + \|\,\mathbf{grad}\,\omega\|_{\infty,T}^2 \sum_{T\subset G} \|\beta_T b_T\|_{0,T}^2\,\big)$$

$$\le C h^2 \|\boldsymbol{\phi}_b\|_{1,G}^2,$$

where we used the fact that

$$\|b_T\|_{0,T} \le C\,h\|b_T\|_{1,T}.$$

Taking $\psi_h = \psi_b + \psi_l \in \mathring{V}_h(G)$ we thus have

$$\|\omega\phi_h - \psi_h\|_{1,\Omega} \le C\,h\big(\,\|\phi_b\|_{1,G} + \|\phi_l\|_{1,G}\,\big).$$

We complete the proof by showing that $\|\phi_b\|_{1,T} + \|\phi_l\|_{1,T} \le C\|\phi_b + \phi_l\|_{1,T}$ for any triangle T with the constant C depending only on the minimum angle of T. Since $\int_T \mathbf{grad}\,\phi_b \cdot \mathbf{grad}\,\phi_l = 0$, it suffices to prove that

$$\|\phi_b\|_{0,T} + \|\phi_l\|_{0,T} \le C\|\phi_b + \phi_l\|_{0,T}.$$

If T is the unit triangle this hold by equivalence of all norms on the finite dimensional space of cubic polynomials, and the extension to an arbitrary triangle is accomplished by scaling.

3.4 Interior Duality Estimates

Let $(\boldsymbol{\phi}, p) \in \boldsymbol{H}^1 \times L^2$ be some solution to the generalized Stokes equations

$$-\boldsymbol{\Delta}\boldsymbol{\phi} + \mathbf{grad}\, p = \boldsymbol{F},$$
$$\operatorname{div} \boldsymbol{\phi} = K.$$

Regardless of the boundary conditions used to specify the particular solution, $(\boldsymbol{\phi}, p)$ satisfies

$$\big(\, \mathbf{grad}\, \boldsymbol{\phi}, \mathbf{grad}\, \boldsymbol{\psi}\,\big) - \big(\operatorname{div} \boldsymbol{\psi}, p\,\big) = \big(\,\boldsymbol{F}, \boldsymbol{\psi}\,\big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1,$$
$$\big(\operatorname{div} \boldsymbol{\phi}, q\,\big) = \big(\,K, q\,\big) \quad \text{for all } q \in L^2.$$

Similarly, regardless of the particular boundary conditions, the finite element solution $(\boldsymbol{\phi}_h, p_h) \in \boldsymbol{V}_h \times W_h$ satisfies

$$\big(\, \mathbf{grad}\, \boldsymbol{\phi}_h, \mathbf{grad}\, \boldsymbol{\psi}\,\big) - \big(\operatorname{div} \boldsymbol{\psi}, p_h\,\big) = \big(\,\boldsymbol{F}, \boldsymbol{\psi}\,\big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h,$$
$$\big(\operatorname{div} \boldsymbol{\phi}_h, q\,\big) = \big(\,K, q\,\big) \quad \text{for all } q \in \mathring{W}_h.$$

Therefore

$$\big(\, \mathbf{grad}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{grad}\, \boldsymbol{\psi}\,\big) - \big(\operatorname{div} \boldsymbol{\psi}, p - p_h\,\big) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{3.4.1}$$

$$\big(\operatorname{div}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), q\,\big) = 0 \quad \text{for all } q \in W_h. \tag{3.4.2}$$

The interior interior error analysis starts from these interior discretization equations.

Theorem 3.4.1. *Let $G_0 \Subset G$ be concentric open disks with closures contained in Ω and s an arbitrary nonnegative integer. Then there exists a constant C such that* *if* $(\phi, p) \in \boldsymbol{H}^1 \times L^2$, *and* $(\phi_h, p_h) \in \boldsymbol{V}_h \times W_h$ *satisfy* (3.4.1) *and* (3.4.2), *we have*

$$\|\phi - \phi_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \leq C\big(h\|\phi - \phi_h\|_{1,G} + h\|p - p_h\|_{0,G} + \|\phi - \phi_h\|_{-s,G} + \|p - p_h\|_{-1-s,G} \big). \tag{3.4.3}$$

In order to prove the theorem we first establish two lemmas.

Lemma 3.4.2. *Under the hypotheses of Theorem 3.4.1, there exists a constant* C *for which*

$$\|p - p_h\|_{-s-1,G_0} \leq C\big(h\|\phi - \phi_h\|_{1,G} + h\|p - p_h\|_{0,G} + \|\phi - \phi_h\|_{-s-1,G} + \|p - p_h\|_{-s-2,G} \big).$$

Proof. Choose a function $\omega \in C_0^\infty(G)$ which is identically 1 on G_0. Also choose a function $\delta \in C_0^\infty(G_0)$ with integral 1. Then

$$\|p - p_h\|_{-s-1,G_0} \leq \|\omega(p - p_h)\|_{-s-1,G} = \sup_{\substack{g \in \mathring{H}^{s+1}(G) \\ g \neq 0}} \frac{\big(\,\omega(p - p_h), g\,\big)}{\|g\|_{s+1,G}}. \tag{3.4.4}$$

Now

$$\big(\,\omega(p - p_h), g\,\big) = \big(\,\omega(p - p_h), g - \delta \int_G g\,\big) + \big(\,\omega(p - p_h), \delta\,\big) \int_G g$$

and clearly

$$\big|\, \big(\,\omega(p - p_h), \delta\,\big) \int_G g \,\big| \leq C\|p - p_h\|_{-s-2,G}\|g\|_{0,G}.$$

Since $g - \delta \int_G g \in H^{s+1}(G) \cap \hat{L}^2(G)$ it follows from Lemma 3.2.1 that there exist $\boldsymbol{\Phi} \in \boldsymbol{H}^{s+2}(G) \cap \mathring{\boldsymbol{H}}^1(G)$ and $P \in H^{s+1}(G) \cap \hat{L}^2(G)$ such that

$$\big(\,\mathbf{grad}\,\boldsymbol{\Phi}, \mathbf{grad}\,\boldsymbol{\psi}\,\big) - \big(\,\operatorname{div}\boldsymbol{\psi}, P\,\big) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(G), \tag{3.4.5}$$

$$\big(\,\operatorname{div}\boldsymbol{\Phi}, q\,\big) = \big(\,g - \delta \int_G g, q\big) \quad \text{for all } q \in L^2(G). \tag{3.4.6}$$

Furthermore,

$$\|\mathbf{\Phi}\|_{s+2,G} + \|P\|_{s+1,G} \le C\|g\|_{s+1,G}. \tag{3.4.7}$$

Then, taking $q = \omega(p - p_h)$ in (3.4.6), we obtain

$$\begin{aligned}
&\big(g - \delta \int_G g, \omega(p - p_h) \big) \\
&= \big(\operatorname{div} \mathbf{\Phi}, \omega(p - p_h) \big) = \big(\operatorname{div}(\omega\mathbf{\Phi}), p - p_h \big) - \big(\mathbf{grad}\,\omega, (p - p_h)\mathbf{\Phi} \big) \\
&= \big(\operatorname{div}(\omega\mathbf{\Phi})^I, p - p_h \big) + \Big\{ \big(\operatorname{div}[\omega\mathbf{\Phi} - (\omega\mathbf{\Phi})^I], p - p_h \big) - \big(\mathbf{grad}\,\omega, (p - p_h)\mathbf{\Phi} \big) \Big\} \\
&=: A_1 + B_1.
\end{aligned} \tag{3.4.8}$$

Here the superscript I is the approximation operator specified in property A1 of section 3.3. Choosing $\boldsymbol{\psi} = (\omega\mathbf{\Phi})^I$ in (3.4.1), we get

$$\begin{aligned}
A_1 &:= \big(\operatorname{div}(\omega\mathbf{\Phi})^I, p - p_h \big) = \big(\mathbf{grad}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{grad}(\omega\mathbf{\Phi})^I \big) \\
&= \big(\mathbf{grad}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{grad}(\omega\mathbf{\Phi}) \big) + \big(\mathbf{grad}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{grad}[(\omega\mathbf{\Phi})^I - (\omega\mathbf{\Phi})] \big) \\
&= \big(\mathbf{grad}[\omega(\boldsymbol{\phi} - \boldsymbol{\phi}_h)], \mathbf{grad}\,\mathbf{\Phi} \big) + \Big\{ \mathrm{R}\big(\omega, \mathbf{\Phi}, \boldsymbol{\phi} - \boldsymbol{\phi}_h \big) \\
&\quad + \big(\mathbf{grad}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{grad}[(\omega\mathbf{\Phi})^I - \omega\mathbf{\Phi}] \big) \Big\} =: A_2 + B_2,
\end{aligned} \tag{3.4.9}$$

where R is defined in (3.2.1). Next, setting $\boldsymbol{\psi} = \omega(\boldsymbol{\phi} - \boldsymbol{\phi}_h)$ in (3.4.5), we obtain

$$\begin{aligned}
A_2 &:= \big(\mathbf{grad}[\omega(\boldsymbol{\phi} - \boldsymbol{\phi}_h)], \mathbf{grad}\,\mathbf{\Phi} \big) = \big(\operatorname{div}[\omega(\boldsymbol{\phi} - \boldsymbol{\phi}_h)], P \big) \\
&= \big(\operatorname{div}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \omega P \big) + \big(\mathbf{grad}\,\omega, P(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \big) \\
&= \big(\operatorname{div}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \omega P - (\omega P)^I \big) + \big(\mathbf{grad}\,\omega, P(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \big),
\end{aligned}$$

where we applied (3.4.2) in the last step.

Applying the approximation property $A1$ and (3.2.2) we get

$$\begin{aligned}
|B_1| &\le C\big(h\|\mathbf{\Phi}\|_{2,G}\|p - p_h\|_{0,G} + \|\mathbf{\Phi}\|_{s+2,G}\|p - p_h\|_{-s-2,G} \big), \\
|B_2| &\le C\big(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-s-1,G}\|\mathbf{\Phi}\|_{s+2,G} + h\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G}\|\mathbf{\Phi}\|_{2,G} \big), \\
|A_2| &\le C\big(h\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G}\|P\|_{1,G} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-s-1}\|P\|_{s+1,G} \big).
\end{aligned} \tag{3.4.10}$$

Substituting (3.4.7) into (3.4.10) and combining the result with (3.4.4), (3.4.8), and (3.4.9), we arrive at (3.4.3). □

Now we state the second lemma to be used in the proof of Theorem 3.4.1.

Lemma 3.4.3. *Under the hypotheses of Theorem 3.4.1, there exists a constant C for which*

$$\|\phi - \phi_h\|_{-s,G_0} \le C\big(\, h\|\phi - \phi_h\|_{1,G} + h\|p - p_h\|_{0,G} \\ + \|\phi - \phi_h\|_{-s-1,G} + \|p - p_h\|_{-s-2,G}\,\big).$$

Proof. Given $\boldsymbol{F} \in \boldsymbol{H}^s(G)$, define $\boldsymbol{\Phi} \in \boldsymbol{H}^{s+2}(G) \cap \mathring{\boldsymbol{H}}^1(G)$ and $P \in H^{s+1}(G) \cap \hat{L}^2(G)$ by

$$\big(\,\mathbf{grad}\,\boldsymbol{\Phi}, \mathbf{grad}\,\boldsymbol{\psi}\,\big) - \big(\,\mathrm{div}\,\boldsymbol{\psi}, P\,\big) = \big(\,\boldsymbol{F}, \boldsymbol{\psi}\,\big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(G), \tag{3.4.13}$$

$$\big(\,\mathrm{div}\;\boldsymbol{\Phi}, q\,\big) = 0 \qquad\qquad \text{for all } q \in L^2(G). \tag{3.4.14}$$

Then, by Lemma 3.2.1,

$$\|\boldsymbol{\Phi}\|_{s+2,G} + \|P\|_{s+1,G} \le\; C\|\boldsymbol{F}\|_{s,G}, \qquad C = C(G_0, G).$$

Now

$$\|\phi - \phi_h\|_{-s,G_0} \le \|\omega(\phi - \phi_h)\|_{-s,G} = \sup_{\substack{\boldsymbol{F}\in\mathring{\boldsymbol{H}}^s(G)\\ \boldsymbol{F}\neq 0}} \frac{\big(\,\omega(\phi - \phi), \boldsymbol{F}\,\big)}{\|\boldsymbol{F}\|_{s,G}}$$

with ω as in the proof of the previous lemma. Setting $\boldsymbol{\psi} = \omega\big(\,\phi - \phi_h\,\big)$ in (3.4.13), we get

$$\begin{aligned}\big(\,\omega(\phi - \phi_h), \boldsymbol{F}\,\big) &= \big(\,\mathbf{grad}\,\boldsymbol{\Phi}, \mathbf{grad}[\omega(\phi - \phi_h)]\,\big) - \big(\,\mathrm{div}[\omega(\phi - \phi_h)], P\,\big) \\ &= \Big\{\big(\,\mathbf{grad}(\omega\boldsymbol{\Phi}), \mathbf{grad}(\phi - \phi_h)\,\big) - \big(\,\mathrm{div}(\phi - \phi_h), \omega P\,\big)\Big\} \\ &\quad - \Big\{\mathrm{R}\big(\,\omega, \boldsymbol{\Phi}, \phi - \phi_h\,\big) + \big(\,\mathbf{grad}\,\omega, P(\phi - \phi_h)\,\big)\Big\} =: E_1 + F_1,\end{aligned}$$

To estimate E_1, we set $q = (\omega P)^I$ in (3.4.2) and obtain

$$E_1 = \big(\,\mathbf{grad}(\omega\mathbf{\Phi})^I, \mathbf{grad}(\phi - \phi_h)\,\big) - \Big\{\big(\,\mathrm{div}(\phi - \phi_h), \omega P - (\omega P)^I\,\big) - \big(\,\mathbf{grad}[\omega\mathbf{\Phi} - (\omega\mathbf{\Phi})^I], \mathbf{grad}(\phi - \phi_h)\,\big)\Big\} =: E_2 + F_2.$$

Taking $\boldsymbol{\psi} = (\omega\mathbf{\Phi})^I$ in (3.4.1), we arrive at

$$\begin{aligned} E_2 &= \big(\,\mathbf{grad}(\omega\mathbf{\Phi})^I, \mathbf{grad}(\phi - \phi_h)\,\big) \\ &= \big(\,\mathrm{div}(\omega\mathbf{\Phi})^I, p - p_h\,\big) \\ &= \big(\,\mathrm{div}(\omega\mathbf{\Phi}), p - p_h\,\big) + \big(\,\mathrm{div}[(\omega\mathbf{\Phi})^I - (\omega\mathbf{\Phi})], p - p_h\,\big) \\ &= \big(\,\mathbf{grad}\,\omega, (p - p_h)\mathbf{\Phi}\,\big) + \big(\,\mathrm{div}[(\omega\mathbf{\Phi})^I - (\omega\mathbf{\Phi})], p - p_h\,\big), \end{aligned}$$

where we applied (3.4.14) in the last step. Applying (3.2.2) and the approximation property $A1$, we have

$$\begin{aligned} |\,F_1\,| &\le C(\|\phi - \phi_h\|_{-s-1,G}\|\mathbf{\Phi}\|_{s+2,G} + \|\phi - \phi_h\|_{-s-1,G}\|P\|_{s+1,G}), \\ |\,F_2\,| &\le Ch(\|\phi - \phi_h\|_{1,G}\|P\|_{1,G} + \|\phi - \phi_h\|_{1,G}\|\mathbf{\Phi}\|_{2,G}), \\ |\,E_2\,| &\le C\big(\,\|p - p_h\|_{-s-2,G}\|\mathbf{\Phi}\|_{s+2,G} + h\,\|p - p_h\|_{0,G}\|\mathbf{\Phi}\|_{2,G}\,\big). \end{aligned}$$

From these bounds we get the desired result. $\square$

Proof of Theorem 3.4.1. Let $G_0 \Subset G_1 \Subset \ldots G_s = G$ be concentric disks. First applying Lemma 3.4.2 and Lemma 3.4.3 with s replaced by 0 and G replaced by G_1, we obtain

$$\|\phi - \phi_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \le C\big(\,h\|\phi - \phi_h\|_{1,G_1} + h\|p - p_h\|_{0,G_1} + \|\phi - \phi_h\|_{-1,G_1} + \|p - p_h\|_{-2,G_1}\,\big).$$

To estimate $\|\phi - \phi_h\|_{-1,G_1}$ and $\|p - p_h\|_{-2,G_1}$, we again apply Lemma 3.4.2 and Lemma 3.4.3, this time with G_0 and G being replaced by G_1 and G_2 and s replaced by 1. Thus, we get

$$\|\phi - \phi_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \le C\big(h\|\phi - \phi_h\|_{1,G_2} + h\|p - p_h\|_{0,G_2} + \|\phi - \phi_h\|_{-2,G_2} + \|p - p_h\|_{-3,G_2} \big).$$

Continuing in this fashion, we obtain (3.4.3). $\square$

3.5 Interior Error Estimates

In this section we state and prove the main result of this chapter, Theorem 3.5.3. First we obtain in Lemma 3.5.1 a bound on solutions of the homogeneous discrete system. In Lemma 3.5.2 this bound is iterated to get a better bound, which is then used to establish the desired local estimate on disks. Finally Theorem 3.5.3 extends this estimate to arbitrary interior domains.

Lemma 3.5.1. *Suppose* $(\phi_h, p_h) \in \boldsymbol{V}_h \times W_h$ *satisfies*

$$(\,\mathbf{grad}\,\phi_h, \mathbf{grad}\,\psi\,) - (\,\operatorname{div}\psi, p_h\,) = 0 \quad \text{for all } \psi \in \mathring{\boldsymbol{V}}_h, \tag{3.5.1}$$

$$(\,\operatorname{div}\phi_h, q\,) = 0 \quad \text{for all } q \in \mathring{W}_h. \tag{3.5.2}$$

Then for any concentric disks $G_0 \Subset G \Subset \Omega$, *and any nonnegative integer* t, *we have*

$$\|\phi_h\|_{1,G_0} + \|p_h\|_{0,G_0} \le C\,\big(\,h\|\phi_h\|_{1,G} + h\|p_h\|_{0,G} + \|\phi_h\|_{-t,G} + \|p_h\|_{-t-1,G}\,\big), \tag{3.5.3}$$

where $C = C(t, G_0, G)$.

Proof. Let G_h, $G_0 \Subset G_h \Subset G$, be as in Assumption A4. Let G' be another disk concentric with G_0 and G, such that $G_0 \Subset G' \Subset G_h$, and construct $\omega \in C_0^\infty(G')$ with $\omega \equiv 1$ on G_0. Set $\widetilde{\phi_h} = \omega\phi_h \in \mathring{\boldsymbol{H}}^1(G')$, $\widetilde{p_h} = \omega p_h \in L^2(G')$. By Lemma 3.3.1, we may define functions $\pi\widetilde{\boldsymbol{\phi}_h} \in \mathring{\boldsymbol{V}}_h(G_h)$ and $\pi\widetilde{p_h} \in W_h(G_h)$ by the equations

$$\left(\,\mathbf{grad}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathbf{grad}\,\boldsymbol{\psi}\,\right) - \left(\,\operatorname{div}\boldsymbol{\psi}, \widetilde{p_h} - \pi\widetilde{p_h}\,\right) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h), \tag{3.5.4}$$

$$\left(\,\operatorname{div}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), q\,\right) = 0 \quad \text{for all } q \in W_h(G_h), \tag{3.5.5}$$

together with $\int_{G_h}(\pi\widetilde{p_h} - \widetilde{p_h}) = 0$. Furthermore, there exists a constant C such that

$$\begin{aligned}
&\|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} \\
&\le C\Big(\inf_{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}_h} - \boldsymbol{\psi}\|_{1,G_h} + \inf_{q\in W_h(G_h)} \|\widetilde{p_h} - q\|_{0,G_h}\,\Big) \\
&\le Ch(\|\boldsymbol{\phi}_h\|_{1,G'} + \|p_h\|_{0,G'}),
\end{aligned} \tag{3.5.6}$$

where we have used the superapproximation property in the last step.

To prove (3.5.3), note that

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} &\le \|\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h}\|_{0,G_h} \\
&\le \|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h} \\
&\le Ch(\|\boldsymbol{\phi}_h\|_{1,G'} + \|p_h\|_{0,G'}) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned} \tag{3.5.7}$$

Next, we bound $\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}$. In (3.5.4) we take $\boldsymbol{\psi} = \pi\widetilde{\boldsymbol{\phi}_h}$ to obtain, for a positive constant c,

$$\begin{aligned}
c\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}^2 &\le \left(\,\mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}\,\right) \\
&= \left(\,\mathbf{grad}\,\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}\,\right) - \left(\,\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \widetilde{p_h} - \pi\widetilde{p_h}\,\right).
\end{aligned} \tag{3.5.8}$$

For the first term on the right hand side of (3.5.8), we have

$$\begin{aligned}
&\left(\,\mathbf{grad}\,\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}\,\right) = \left(\,\mathbf{grad}(\omega\boldsymbol{\phi}_h), \mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}\,\right) \\
&= \left(\,\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}(\omega\pi\widetilde{\boldsymbol{\phi}_h})\,\right) - \mathrm{R}\left(\,\omega, \pi\widetilde{\boldsymbol{\phi}_h}, \boldsymbol{\phi}_h\,\right) \\
&= \left(\,\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I\,\right) + \Big\{\left(\,\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}[\omega\pi\widetilde{\boldsymbol{\phi}_h} - (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I]\,\right) \\
&\quad -\mathrm{R}\left(\,\omega, \pi\widetilde{\boldsymbol{\phi}_h}, \boldsymbol{\phi}_h\,\right)\Big\} =: G_1 + H_1.
\end{aligned} \tag{3.5.9}$$

To bound G_1, we take $\boldsymbol{\psi} = (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I$ in (3.5.1) and get

$$\begin{aligned}
G_1 &= \big(\operatorname{div}(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I, p_h\big)\\
&= \big(\operatorname{div}(\omega\pi\widetilde{\boldsymbol{\phi}_h}), p_h\big) + \big(\operatorname{div}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I - \omega\pi\widetilde{\boldsymbol{\phi}_h}], p_h\big)\\
&= \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \omega p_h\big) + \big(\mathbf{grad}\,\omega, p_h\pi\widetilde{\boldsymbol{\phi}_h}\big) + \big(\operatorname{div}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I - \omega\pi\widetilde{\boldsymbol{\phi}_h}], p_h\big)\\
&= \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \widetilde{p_h}\big) + \big(\mathbf{grad}\,\omega, p_h\pi\widetilde{\boldsymbol{\phi}_h}\big) + \big(\operatorname{div}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I - \omega\pi\widetilde{\boldsymbol{\phi}_h}], p_h\big)\\
&=: \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \widetilde{p_h}\big) + H_2.
\end{aligned} \tag{3.5.10}$$

Combining (3.5.7), (3.5.8), (3.5.9), and (3.5.10), we obtain

$$\begin{aligned}
c\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}^2 &\le \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \widetilde{p_h}\big) + H_1 + H_2 - \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \widetilde{p_h} - \pi\widetilde{p_h}\big)\\
&= \big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \pi\widetilde{p_h}\big) + H_1 + H_2.
\end{aligned} \tag{3.5.11}$$

Taking $q = \pi\widetilde{p_h}$ in (3.5.5), we get

$$\begin{aligned}
\big(\operatorname{div}\pi\widetilde{\boldsymbol{\phi}_h}, \pi\widetilde{p_h}\big) &= \big(\operatorname{div}\widetilde{\boldsymbol{\phi}_h}, \pi\widetilde{p_h}\big) = \big(\operatorname{div}(\omega\boldsymbol{\phi}_h), \pi\widetilde{p_h}\big)\\
&= \big(\operatorname{div}\boldsymbol{\phi}_h, \omega\pi\widetilde{p_h}\big) + \big(\mathbf{grad}\,\omega, \pi\widetilde{p_h}\boldsymbol{\phi}_h\big)\\
&= \big(\operatorname{div}\boldsymbol{\phi}_h, \omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I\big) + \big(\mathbf{grad}\,\omega, \pi\widetilde{p_h}\boldsymbol{\phi}_h\big) =: H_3,
\end{aligned} \tag{3.5.12}$$

where we used (3.5.2) at the last step. Applying the Schwartz inequality, (3.2.2), and the superapproximation property A2, we get

$$\begin{aligned}
|\, H_1 \,| &\le C\big(h\,\|\boldsymbol{\phi}_h\|_{1,G'} + \|\boldsymbol{\phi}_h\|_{0,G'}\big)\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h},\\
|\, H_2 \,| &\le C\big(\|p_h\|_{-1,G'} + h\,\|p_h\|_{0,G'}\big)\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h},\\
|\, H_3 \,| &\le C\big(h\,\|\boldsymbol{\phi}_h\|_{1,G'} + \|\boldsymbol{\phi}_h\|_{0,G'}\big)\|\pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned}$$

Combining the above three inequalities with (3.5.11) and (3.5.12), and using the arithmetic-geometry mean inequality, we arrive at

$$\begin{aligned}
\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}^2 &\le C_1\big(h^2\|\boldsymbol{\phi}_h\|_{1,G'}^2 + \|\boldsymbol{\phi}_h\|_{0,G'}^2 + h^2\|p_h\|_{0,G'}^2 + \|p_h\|_{-1,G'}^2\big)\\
&\quad + C_2\big(\|\boldsymbol{\phi}_h\|_{0,G'} + h\,\|\boldsymbol{\phi}_h\|_{1,G'}\big)\|\pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned} \tag{3.5.13}$$

Next we estimate $\|\pi\widetilde{p_h}\|_{0,G_h}$. By the triangle inequality,

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le \left\|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\operatorname{meas}(G_h)}\right\|_{0,G_h} + \operatorname{meas}(G_h)^{-1}\left\|\int_{G_h}(\pi\widetilde{p_h} - \widetilde{p_h})\right\|_{0,G_h}$$
$$+ \operatorname{meas}(G_h)^{-1}\left\|\int_{G_h}\widetilde{p_h}\right\|_{0,G_h}. \tag{3.5.14}$$

Notice that the second term on the right hand side of (3.5.14) is bounded above by the right hand side of (3.5.6), and, for the last term,

$$\left\|\int_{G_h}\widetilde{p_h}\right\|_{0,G_h} = \left\|\int_{G_h}\omega p_h\right\|_{0,G_h} \le C\|p_h\|_{-1,G'}. \tag{3.5.15}$$

To estimate the first term, we use the inf-sup condition,

$$\left\|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\operatorname{meas}(G_h)}\right\|_{0,G_h} \le C \sup_{\substack{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h(G_h)\\ \boldsymbol{\psi}\ne 0}} \frac{\left(\operatorname{div}\boldsymbol{\psi}, \pi\widetilde{p_h}\right)_{G_h}}{\|\boldsymbol{\psi}\|_{1,G_h}}. \tag{3.5.16}$$

To deal with the numerator on the right hand side of (3.5.16), we apply (3.5.4),

$$\begin{aligned}
\left(\operatorname{div}\boldsymbol{\psi}, \pi\widetilde{p_h}\right) &= \left(\operatorname{div}\boldsymbol{\psi}, \widetilde{p_h}\right) - \left(\mathbf{grad}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathbf{grad}\,\boldsymbol{\psi}\right)\\
&= \left(\operatorname{div}\boldsymbol{\psi}, \omega p_h\right) - \left(\mathbf{grad}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathbf{grad}\,\boldsymbol{\psi}\right)\\
&= \left(\operatorname{div}(\omega\boldsymbol{\psi}), p_h\right) - \left(\mathbf{grad}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathbf{grad}\,\boldsymbol{\psi}\right) - \left(\mathbf{grad}\,\omega, p_h\boldsymbol{\psi}\right)\\
&= \left(\operatorname{div}(\omega\boldsymbol{\psi})^I, p_h\right) - \left(\mathbf{grad}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathbf{grad}\,\boldsymbol{\psi}\right)\\
&\quad + \left(\operatorname{div}(\omega\boldsymbol{\psi} - (\omega\boldsymbol{\psi})^I), p_h\right) - \left(\mathbf{grad}\,\omega, p_h\boldsymbol{\psi}\right).
\end{aligned} \tag{3.5.17}$$

We use (3.5.1) to attack $\left(\operatorname{div}(\omega\boldsymbol{\psi})^I, p_h\right)$ and get

$$\begin{aligned}
&\left(\operatorname{div}(\omega\boldsymbol{\psi})^I, p_h\right) = \left(\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}(\omega\boldsymbol{\psi})^I\right)\\
&= \left(\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}(\omega\boldsymbol{\psi})\right) + \left(\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}]\right)\\
&= \left(\mathbf{grad}(\omega\boldsymbol{\phi}_h), \mathbf{grad}\,\boldsymbol{\psi}\right) + \left\{\mathrm{R}\left(\omega, \boldsymbol{\psi}, \boldsymbol{\phi}_h\right) + \left(\mathbf{grad}\,\boldsymbol{\phi}_h, \mathbf{grad}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}]\right)\right\}\\
&=: \left(\mathbf{grad}\,\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\boldsymbol{\psi}\right) + M_1.
\end{aligned} \tag{3.5.18}$$

Combining (3.5.17) and (3.5.18), we get

$$\begin{aligned}\left(\operatorname{div}\boldsymbol{\psi}, \pi\widetilde{p_h}\right) &= \left\{\left(\operatorname{div}(w\boldsymbol{\psi} - (w\boldsymbol{\psi})^I), p_h\right) - \left(\mathbf{grad}\, w, p_h\boldsymbol{\psi}\right)\right\} \\ &\quad + \left(\mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\boldsymbol{\psi}\right) + M_1 \\ &=: \left(\mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\boldsymbol{\psi}\right) + M_1 + M_2. \end{aligned} \tag{3.5.19}$$

Then applying the superapproximation property, the Schwartz inequality, and (3.2.2), we arrive at

$$\begin{aligned}| M_1 | &\le C\left(\|\boldsymbol{\phi}_h\|_{0,G'} + h\,\|\boldsymbol{\phi}_h\|_{1,G'}\right)\|\boldsymbol{\psi}\|_{1,G_h}, \\ | M_2 | &\le C\left(h\,\|p_h\|_{0,G'} + \|p_h\|_{-1,G'}\right)\|\boldsymbol{\psi}\|_{1,G_h}, \\ \left|\left(\mathbf{grad}\,\pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{grad}\,\boldsymbol{\psi}\right)\right| &\le \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}\|\boldsymbol{\psi}\|_{1,G_h}.\end{aligned}$$

Combining (3.5.14), (3.5.15) , (3.5.16), and (3.5.19) with the above three inequalities, we obtain

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le C\left(h\,\|\boldsymbol{\phi}_h\|_{1,G'} + \|\boldsymbol{\phi}_h\|_{0,G'} + h\,\|p_h\|_{0,G'} + \|p_h\|_{-1,G'} + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}\right). \tag{3.5.20}$$

Substituting (3.5.20) into (3.5.13), we obtain

$$\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} \le C\left(h\,\|\boldsymbol{\phi}_h\|_{1,G'} + \|\boldsymbol{\phi}_h\|_{0,G'} + h\,\|p_h\|_{0,G'} + \|p_h\|_{-1,G'}\right). \tag{3.5.21}$$

Thus, substituting (3.5.21) back into (3.5.20), we find that $\|\pi\widetilde{p_h}\|_{0,G_h}$ is also bounded above by the right hand side of (3.5.21). Therefore, from (3.5.7) we obtain

$$\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \le C\left(h\,\|\boldsymbol{\phi}_h\|_{1,G'} + \|\boldsymbol{\phi}_h\|_{0,G'} + h\,\|p_h\|_{0,G'} + \|p_h\|_{-1,G'}\right).$$

Applying Theorem 3.4.1 for the case that $\boldsymbol{\phi} = p = 0$ and G' in place of G_0, we finally arrive at

$$\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \le C\left(h\,\|\boldsymbol{\phi}_h\|_{1,G} + \|\boldsymbol{\phi}_h\|_{-t,G} + h\,\|p_h\|_{0,G} + \|p_h\|_{-t-1,G}\right). \quad \square$$

Lemma 3.5.2. *Suppose the conditions of Lemma 3.5.1 are satisfied. Then*

$$\|\phi_h\|_{1,G_0} + \|p_h\|_{0,G_0} \le C\big(\|\phi_h\|_{-t,G} + \|p_h\|_{-t-1,G}\big). \tag{3.5.22}$$

Proof. Let $G_0 \Subset G_1 \Subset \ldots \Subset G_{t+2} = G$ be concentric disks and apply Lemma 3.5.1 to each pair $G_j \Subset G_{j+1}$ to get

$$\begin{aligned}&\|\phi_h\|_{1,G_j} + \|p_h\|_{0,G_j}\\ &\le C\,\big(h\|\phi_h\|_{1,G_{j+1}} + h\|p_h\|_{0,G_{j+1}} + \|\phi_h\|_{-t,G_{j+1}} + \|p_h\|_{-t-1,G_{j+1}}\big).\end{aligned} \tag{3.5.23}$$

Combining these we obtain

$$\begin{aligned}\|\phi_h\|_{1,G_0} + \|p_h\|_{0,G_0} \le C\,\big(&h^{t+1}\|\phi_h\|_{1,G_{t+1}} + h^{t+1}\|p_h\|_{0,G_{t+1}}\\ &+ \|\phi_h\|_{-t,G_{t+1}} + \|p_h\|_{-t+1,G_{t+1}}\big).\end{aligned} \tag{3.5.24}$$

While by A3, we can find G_h, $G_{t+1} \Subset G_h \Subset G_{t+2} = G$, such that

$$\begin{aligned}h^{t+1}\|\phi_h\|_{1,G_{t+1}} &\le h^{t+1}\|\phi_h\|_{1,G_h} \le C\,\|\phi_h\|_{-t,G_h} \le C\,\|\phi_h\|_{-t,G},\\ h^{t+1}\|p_h\|_{0,G_{t+1}} &\le h^{t+1}\|p_h\|_{0,G_h} \le C\,\|p_h\|_{-t-1,G_h} \le C\,\|p_h\|_{-t-1,G}.\end{aligned} \tag{3.5.25}$$

Thus inequality (3.5.22) follows from (3.5.23), (3.5.24), and (3.5.25). □

We now state the main result of the chapter.

Theorem 3.5.3. *Let $\Omega_0 \Subset \Omega_1 \Subset \Omega$ and suppose that $(\phi, p) \in \boldsymbol{H}^1 \times L^2$ (the exact solution) satisfies $\phi|_{\Omega_1} \in \boldsymbol{H}^m(\Omega_1)$ and $p|_{\Omega_1} \in \boldsymbol{H}^{m-1}(\Omega_1)$ for some integer $m > 0$. Suppose that $(\phi_h,\, p_h) \in \boldsymbol{V}_h \times W_h$ (the finite element solution) is given so that* (3.4.1) *and* (3.4.2) *hold. Let t be a nonnegative integer. Then there exists a constant C depending only on Ω_1, Ω_0, and t, such that*

$$\begin{aligned}\|\phi - \phi_h\|_{s,\Omega_0} + \|p - p_h\|_{s-1,\Omega_0} \le C\,\big(&h^{r_1-s}\|\phi\|_{m,\Omega_1} + h^{r_2-s}\|p\|_{m-1,\Omega_1}\\ &+ \|\phi - \phi_h\|_{-t,\Omega_1} + \|p - p_h\|_{-t-1,\Omega_1}\big), \quad s = 0, 1\end{aligned} \tag{3.5.26}$$

with $r_1 = \min(k_1+1, m)$, $r_2 = \min(k_2+2, m)$, *and* k_1, k_2 *as in A1.*

The theorem will follow easily from a slightly more localized version.

Theorem 3.5.4. *Suppose the hypotheses of Theorem 3.5.3 are fulfilled and, in addition, that* $\Omega_0 = G_0$ *and* $\Omega_1 = G_1$ *are concentric disks. Then the conclusion of the theorem holds.*

Proof. Let $G_0' \Subset G'$ be further concentric disks strictly contained between G_0 and G and let G_h be a domain strictly contained between G' and G for which properties A3 and A4 hold. Thus

$$G_0 \Subset G_0' \Subset G' \Subset G_h \Subset G \Subset \Omega.$$

Take $\omega \in C_0^\infty(G')$ identically 1 on G_0' and set $\widetilde{\boldsymbol{\phi}} = \omega\boldsymbol{\phi}$, $\widetilde{p} = \omega p$. Let $\pi\widetilde{\boldsymbol{\phi}} \in \mathring{\boldsymbol{V}}_h(G)$, $\pi\widetilde{p} \in W_h(G)$ be defined by

$$\big(\,\mathbf{grad}(\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}), \mathbf{grad}\,\boldsymbol{\psi}\,\big) - \big(\,\operatorname{div}\boldsymbol{\psi}, \widetilde{p} - \pi\widetilde{p}\,\big) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h), \tag{3.5.27}$$

$$\big(\,\operatorname{div}(\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}), q\,\big) = 0 \quad \text{for all } q \in W_h(G_h), \tag{3.5.28}$$

together with $\int_{G_h} \pi\widetilde{p} = \int_{G_h} \widetilde{p}$. Then using Lemma 3.3.1 and A1 we have

$$\begin{aligned}
&\|\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_h} + \|\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} \\
&\le C\,\Big(\inf_{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}} - \boldsymbol{\psi}\|_{1,G_h} + \inf_{q\in W_h(G_h)} \|\widetilde{p} - q\|_{0,G_h}\Big) \\
&\le C\,\big(\,h^{r_1-1}\|\boldsymbol{\phi}\|_{m,G_h} + h^{r_2-1}\|p\|_{m-1,G_h}\,\big).
\end{aligned} \tag{3.5.29}$$

Let us now estimate $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0}$ and $\|p - p_h\|_{0,G_0}$. First, the triangle inequality gives us

$$
\begin{aligned}
&\|\phi-\phi_h\|_{1,G_0}+\|p-p_h\|_{0,G_0}\\
&\le \|\phi-\pi\widetilde{\phi}\|_{1,G_0}+\|p-\pi\widetilde{p}\|_{0,G_0}+\|\pi\widetilde{\phi}-\phi_h\|_{1,G_0}+\|\pi\widetilde{p}-p_h\|_{0,G_0}\\
&\le \|\widetilde{\phi}-\pi\widetilde{\phi}\|_{1,G_h}+\|\widetilde{p}-\pi\widetilde{p}\|_{0,G_h}+\|\pi\widetilde{\phi}-\phi_h\|_{1,G_0}+\|\pi\widetilde{p}-p_h\|_{0,G_0}\\
&\le C\left(h^{r_1-1}\|\phi\|_{m,G_h}+h^{r_2-1}\|p\|_{m-1,G_h}\right)+\|\pi\widetilde{\phi}-\phi_h\|_{1,G_0}+\|\pi\widetilde{p}-p_h\|_{0,G_0}.
\end{aligned}
\tag{3.5.30}
$$

From (3.5.27), (3.5.28) and (3.4.1), (3.4.2) we find

$$
\begin{aligned}
\left(\mathbf{grad}(\phi_h-\pi\widetilde{\phi}),\mathbf{grad}\,\psi\right)-\left(\operatorname{div}\psi,p_h-\pi\widetilde{p}\right)&=0\quad\text{for all }\psi\in\mathring{\boldsymbol{V}}_h(G_0'),\\
\left(\operatorname{div}(\phi_h-\pi\widetilde{\phi}),q\right)&=0\quad\text{for all }q\in\mathring{W}_h(G_0').
\end{aligned}
$$

We next apply Lemma 3.5.2 to $\phi_h-\pi\widetilde{\phi}$ and $p_h-\pi\widetilde{p}$ with G replaced by G_0'. Then it follows from (3.5.22) that

$$
\begin{aligned}
&\|\phi_h-\pi\widetilde{\phi}\|_{1,G_0}+\|p_h-\pi\widetilde{p}\|_{0,G_0}\le C\left(\|\phi_h-\pi\widetilde{\phi}\|_{-t,G_0'}+\|p_h-\pi\widetilde{p}\|_{-t-1,G_0'}\right)\\
&\le C\left(\|\phi-\phi_h\|_{-t,G_0'}+\|p-p_h\|_{-t-1,G_0'}+\|\phi-\pi\widetilde{\phi}\|_{-t,G_0'}+\|p-\pi\widetilde{p}\|_{-t-1,G_0'}\right)\\
&\le C\left(\|\phi-\phi_h\|_{-t,G}+\|p-p_h\|_{-t-1,G}+\|\widetilde{\phi}-\pi\widetilde{\phi}\|_{1,G_h}+\|\widetilde{p}-\pi\widetilde{p}\|_{0,G_h}\right).
\end{aligned}
$$

In the light of (3.5.30), (3.5.29), and the above inequality, we have

$$
\begin{aligned}
\|\phi-\phi_h\|_{1,G_0}+\|p-p_h\|_{0,G_0}\le\big(&h^{r_1-1}\|\phi\|_{m,G}+h^{r_2-1}\|p\|_{m-1,G}\\
&+\|\phi-\phi_h\|_{-t,G}+\|p-p_h\|_{-t-1,G}\big).
\end{aligned}
\tag{3.5.31}
$$

Thus, we have proved the desired result for $s=1$. For $s=0$, we just apply Theorem 3.4.1 to the disks G_0 and G' and get

$$
\begin{aligned}
\|\phi-\phi_h\|_{0,G_0}+\|p-p_h\|_{-1,G_0}\le C\big(&h\|\phi-\phi_h\|_{1,G'}+h\|p-p_h\|_{0,G'}\\
&+\|\phi-\phi_h\|_{-t,G'}+\|p-p_h\|_{-t-1,G'}\big).
\end{aligned}
$$

Then, applying (3.5.31) with G_0 replaced by G', we obtain the desired result

$$\|\phi - \phi_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \le C\,\big(\, h^{r_1}\|\phi\|_{m,G} + h^{r_2}\|p\|_{m-1,G} + \|\phi - \phi_h\|_{-t,G} + \|p - p_h\|_{-t-1,G}\,\big). \quad \square$$

Proof of Theorem 3.5.3. The argument here is same as in Theorem 5.1 of [33]. Let $d = d_0/2$ where $d_0 = \mathrm{dist}(\bar{\Omega}_0, \partial\Omega_1)$. Cover $\bar{\Omega}_0$ with a finite number of disks $G_0(x_i)$, $i = 1, 2, \ldots, m$ centered at $x_i \in \bar{\Omega}_0$ with $\operatorname{diam} G_0(x_i) = d$. Let $G(x_i)$, $i = 1, 2, \ldots, k$ be corresponding concentric disks with $\operatorname{diam} G(x_i) = 2d$. Applying Theorem 3.5.4, we have

$$\|\phi - \phi_h\|_{s,G_0(x_i)} + \|p - p_h\|_{s-1,G_0(x_i)} \le C_i\,\big(\, h^{r_1-s}\|\phi\|_{m,G(x_i)} + h^{r_2-s}\|p\|_{m-1,G(x_i)} + \|\phi - \phi_h\|_{-t,G(x_i)} + \|p - p_h\|_{-t-1,G(x_i)}\,\big). \tag{3.5.32}$$

Then the inequality (3.5.26) follows by summing (3.5.32) for every i. $\square$

3.6 An Example Application

As an example, we apply our general result to the Stokes system when the domain is a non-convex polygon, in which case the finite element approximation does not achieve the optimal convergence rate in the energy norm on the whole domain, due to the boundary singularity of the exact solution.

Assume that Ω is a non-convex polygon. Then it is known that the solution of the Stokes system satisfies

$$\phi \in \boldsymbol{H}^{s+1} \cap \boldsymbol{H}_0^1\,,\ p \in H^s,$$
$$\phi \in \boldsymbol{H}^2(\Omega_1),\ p \in H^1(\Omega_1), \quad \text{if} \quad \Omega_1 \Subset \Omega,$$

for $s < s_\Omega$, where s_Ω is a constant which is determined by the largest interior angle of Ω (cf. [19]). For a non-convex polygonal domain we have $1/2 < s_\Omega < 1$. The value of s_Ω for various angles have been tabulated in [19]. For example, for an L-shaped domain, $s_\Omega \sim 0.544$.

The MINI element was introduced by Arnold, Brezzi and Fortin in [2] as a stable Stokes element with few degrees of freedom. Here the velocity is approximated by the space of continuous piecewise linear functions and bubble functions and the pressure is approximated by the space of continuous piecewise linear functions only. Globally we have

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|p - p_h\|_0 \le Ch^s\big(\|\boldsymbol{\phi}\|_{s+1} + \|p\|_s\big),$$

which reflects a loss of accuracy due to the singularity of the solutions.

In order to apply Theorem 3.5.3, we note that a standard duality argument as in [2] gives us

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|p - p_h\|_{-1} \le Ch^{2s}\|\boldsymbol{F}\|_0.$$

Hence, according to Theorem 3.5.3, for $\Omega_0 \Subset \Omega_1 \Subset \Omega$, we have

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} \le C\big(h\|\boldsymbol{\phi}\|_{2,\Omega_1} + h\|p\|_{1,\Omega_1} + h^{2s}\,\|\boldsymbol{F}\|_0\big).$$

Since $2s > 1$, the finite element approximation achieves the optimal order of convergence rate in the energy norm in interior subdomains.

CHAPTER 4

INTERIOR ESTIMATES FOR A FINITE ELEMENT METHOD FOR THE REISSNER-MINDLIN PLATE MODEL

4.1 Introduction

The Reissner-Mindlin plate model describes deformation of a plate with small to moderate thickness subject to a transverse load. The finite element method for this model was studied extensively (cf. [11], [30], and references therein) and it has been known for a long time that a direct application of standard finite element methods usually leads to unreasonly small solution, as the plate thickness approaches zero. This is usually called the "locking" phenomenon of the finite element method for the Reissner-Mindlin plate [11], [30].

The reason behind the locking phenomenon is well known: as the plate thickness becomes very small, the numerical scheme tries to enforce a discrete version of the Kirchoff constraint on the displacement and the rotation fiber normal to the midplane. If the finite element spaces for those two quantities are not chosen wisely, then, together with boundary conditions, the numerical solution reduces to the trivial solution.

Another difficulty relating to the Reissner-Mindlin plate model is that the solution possesses boundary layers, having the plate thickness as the singular parameter. As usual, the strength of the boundary layer is sensitive to the boundary condition. The structure of the dependence of the solution on the plate thickness was analyzed in detail by Arnold and Falk [4], [5].

The purpose of this chapter is to obtain the interior error estimate for the Arnold-Falk element [3] for the Reissner-Mindlin plate model. This element is the first to achieve a locking-free first order (optimal) convergence for the Reissner-Mindlin plate (under the hard clamped boundary condition). However, it does not retain the same order of convergence rate for the plate under the soft simply supported boundary condition, due to a stronger boundary layer effect. By applying the interior estimate to the soft simply supported plate, we are able to obtain the interior convergence rate of the Arnold-Falk element and show that it still possesses (almost) first order convergence rate in the region away from the boundary.

The construction of the Arnold-Falk element is based on an equivalence between the plate equations and an uncoupled system of two Poisson equations plus a Stokes-like system [3]. Arnold and Falk used the nonconforming linear element for the Poisson equation and the MINI element for the Stokes-like system. So the (global or interior) analysis of the Arnold-Falk element consists of two parts: one for the nonconforming method for the Poisson equation and another for the MINI element for the Stokes-like system. Recall that in Chapter 2 we obtained interior estimates for the nonconforming element for the Poisson equation. So the task here is essentially to analyze the interior error estimate of the MINI element for the Stokes-like system.

The organization of chapter is as follows. Section 4.2 presents the Reissner-Mindlin plate equations and its reformulation under the Helmholtz decomposition for the shear stress. The interior regularity of the solution of the singularly perturbed system is studied in section 4.3. The Arnold-Falk element is introduced in section 4.4. Section 4.5 is devoted to the interior duality analysis of the variant of the Stokes system. In section 4.6 we first obtain the interior estimate of the MINI element (Theorem 4.6.2) for the Stokes-like system with perturbation and then use it to get the interior estimate of the Arnold-Falk element for the Reissner-Mindlin plate model (Theorem 4.6.3), which is the main result of the chapter. As an application of the general theory we develop, we consider the soft simply supported plate in section 4.7. We will show that globally, the Arnold-Falk element only achieves (almost) $h^{1/2}$ order convergence for the rotation (Theorem 4.7.3), but away from the boundary layer, (almost) optimal order convergence rate can be obtained (Theorem 4.7.4). Finally, numerical results are shown in section 4.8 which confirm the theoretical predictions.

4.2 Notations and the Reissner–Mindlin Plate Model

The following operators are standard.

$$\mathbf{div}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \partial t_{11}/\partial x + \partial t_{12}/\partial y \\ \partial t_{21}/\partial x + \partial t_{22}/\partial y \end{pmatrix},$$

$$\mathbf{curl}\, p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix}, \quad \operatorname{rot}\boldsymbol{\phi} = \partial\phi_1/\partial y - \partial\phi_2/\partial x.$$

Let Ω denote the region in $\mathbb{R}^2$ occupied by the midsection of the plate, and denote by w and $\boldsymbol{\phi}$ the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. Under the soft simply supported boundary condition, the Reissner-Mindlin plate model determines $(w, \boldsymbol{\phi})$ as the unique solution to the following variational problem:

Find $(w, \boldsymbol{\phi}) \in \mathring{H}^1 \times \boldsymbol{H}^1$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \mathbf{grad}\, w, \boldsymbol{\psi} - \mathbf{grad}\, \mu) = (g, \mu) \quad \text{for all } (\mu, \boldsymbol{\psi}) \in \mathring{H}^1 \times \boldsymbol{H}^1. \quad (4.2.1)$$

Here g is the scaled transverse loading function, t the plate thickness, $\lambda = E\kappa/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, and κ the shear correction factor. The bilinear form a is

$$\begin{aligned} a(\boldsymbol{\phi},\boldsymbol{\psi}) =& \frac{E}{12(1-\nu^2)}\int_\Omega \left(\frac{\partial\phi_1}{\partial x}+\nu\frac{\partial\phi_2}{\partial y}\right)\frac{\partial\psi_1}{\partial x}+\left(\nu\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial y}\right)\frac{\partial\psi_2}{\partial y} \\ &+\frac{1-\nu}{2}\left(\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial x}\right)\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right) \\ =& \int_\Omega \mathcal{C}\mathcal{E}(\boldsymbol{\phi}):\mathcal{E}(\boldsymbol{\psi}). \end{aligned}$$

Here, $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$ and $\mathcal{C}$ is a fourth order tensor defined by the bilinear form a.

Following Brezzi and Fortin [15], equation (4.2.1) can be reformulated by using the Helmholtz Theorem to decompose the shear strain vector

$$\lambda t^{-2}(\mathbf{grad}\, w-\boldsymbol{\phi}) = \mathbf{grad}\, r+\mathbf{curl}\, p, \tag{4.2.2}$$

with $(r,p)\in \mathring{H}^1\times\hat{H}^1$.

Equation (4.2.1) now becomes

Find $(r,\boldsymbol{\phi},p,w)\in \mathring{H}^1\times\boldsymbol{H}^1\times\hat{H}^1\times\mathring{H}^1$ such that

$$(\mathbf{grad}\, r,\mathbf{grad}\,\mu) = (g,\mu) \quad \text{for all } \mu\in\mathring{H}^1, \tag{4.2.3}$$

$$\big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}),\mathcal{E}(\boldsymbol{\psi})\big)-\big(\mathbf{curl}\, p,\boldsymbol{\psi}\big) = \big(\mathbf{grad}\, r,\boldsymbol{\psi}\big) \quad \text{for all } \boldsymbol{\psi}\in\boldsymbol{H}^1, \tag{4.2.4}$$

$$-(\boldsymbol{\phi},\mathbf{curl}\, q)-\lambda^{-1}t^2(\mathbf{curl}\, p,\mathbf{curl}\, q) = 0 \quad \text{for all } q\in\hat{H}^1, \tag{4.2.5}$$

$$(\mathbf{grad}\, w,\mathbf{grad}\, s) = (\boldsymbol{\phi}+\lambda^{-1}t^2\,\mathbf{grad}\, r,\mathbf{grad}\, s) \quad \text{for all } s\in\mathring{H}^1. \tag{4.2.6}$$

Obviously the function r in (4.2.3) is independent of t and the functions $\boldsymbol{\phi}$, p, and w are not. It has been shown in [5] that the transverse displacement w does not suffer from the boundary layer effect under all boundary conditions. However, the regularity of solution $(\boldsymbol{\phi},p)$ for system (4.2.4) and (4.2.5) depends on the boundary condition imposed on the plate. For example, under the hard clamped boundary condition (then, $\boldsymbol{\phi}$ is to be found in space $\overset{\circ}{\boldsymbol{H}}{}^1$, rather than $\boldsymbol{H}^1$), the following holds [3]

$$\|\boldsymbol{\phi}\|_2 + \|p\|_1 \leq C\|g\|_0,$$

with the constant C independent of the plate thickness t. This guarantees the MINI element to achieve a locking free first order convergence rate for the system (4.2.4) and (4.2.5) [3].

But the above estimate does not hold for the soft simply supported plate. In this case, one can only expect that the $\boldsymbol{H}^{3/2}$ norm of function $\boldsymbol{\phi}$ and the $H^{1/2}$ norm of function p to be bounded above, independent of the small parameter t [5]. This is obviously not enough for the finite element method to achieve first order convergence rate. It is also easy to see that a complete understanding of the dependence of the regularity of the solution on the small parameter t is of crucial importance for the convergence analysis of the finite element method. However, for the purpose of interior estimates, we need only know the inteiror regularity of the solution of the Stokes-like system. This will be given in the next section.

In the following, we introduce some notations that will be used in the interior estimate.

Let G be an open subset of Ω, $\omega \in C_0^\infty(G)$, and s an integer. For $\mathbf{\Phi} \in \boldsymbol{H}^s(G)$, $\mathbf{\Psi} \in \boldsymbol{H}^{-s+2}(G)$, $P \in H^s(G)$, and $Q \in H^{-s+2}(G)$, define

$$\mathrm{R}(\omega, \mathbf{\Phi}, \mathbf{\Psi}) = \big(\, \mathcal{C}\mathcal{E}(\omega\mathbf{\Phi}), \mathcal{E}(\mathbf{\Psi})\,\big) - \big(\, \mathcal{C}\mathcal{E}(\mathbf{\Psi}), \mathcal{E}(\omega\mathbf{\Psi})\,\big)$$

and

$$\mathrm{R}'\big(\,\omega, P, Q\,\big) = \big(\,\mathbf{curl}(\omega P), \mathbf{curl}\, Q\,\big) - \big(\,\mathbf{curl}\, P, \mathbf{curl}(\omega Q)\,\big).$$

Then

$$|\mathrm{R}\big(\,\omega, \mathbf{\Phi}, \mathbf{\Psi}\,\big) \leq C\|\mathbf{\Phi}\|_{t,G}\|\mathbf{\Psi}\|_{-t+1,G} \tag{4.2.7}$$

and

$$|\mathrm{R}'(\omega, P, Q) \le C\|P\|_{t,G}\|Q\|_{-t+1,G}, \qquad (4.2.8)$$

for non-negative integers $t \le s$.

4.3 An Interior Regularity Result

In this section we present an interior regularity result for the solution of the singularly perturbed Stokes-like system under the homogeneous Dirichlet boundary condition. We will show that the regularity of the solution in the interior region is not affected by the boundary layer. This will be used in section 4.5 for the the interior duality analysis of the MINI element.

The proof basically follows that in [3, Theorem 7.1] for proving the regularity of the solution of the hard-clamped plate and uses the standard approach for analyzing interior regularities for solutions of elliptic equations.

Theorem 4.3.1. *Let* $\boldsymbol{F} \in \boldsymbol{H}^s(G)$ *and* $K \in H^{s+1}(G) \cap \hat{L}^2(G)$*, where integer* $s \ge 0$ *and* G *is a disk. Then there exists a unique solution* $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^{s+2}(G) \cap \mathring{\boldsymbol{H}}^1(G) \times H^{s+1}(G) \cap \hat{L}^2(G)$ *such that*

$$(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\mathbf{curl}\, P, \boldsymbol{\Psi}) = (\boldsymbol{\Psi}, \boldsymbol{F}) \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G), \qquad (4.3.1)$$

$$-(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - \lambda^{-1} t^2 (\mathbf{curl}\, Q, \mathbf{curl}\, P) = (Q, K) \quad \text{for all } Q \in H^1(G). \qquad (4.3.2)$$

Moreover,

$$\|\boldsymbol{\Phi}\|_{2,G} + \|P\|_{1,G} + t\|P\|_{2,G} + t^2\|P\|_{3,G} \le C(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}), \qquad (4.3.3)$$

$$\|\boldsymbol{\Phi}\|_{s+2,G_0} + \|P\|_{s+1,G_0} + t\|P\|_{s+2,G_0} + t^2\|P\|_{s+3,G_0} \le C(\|\boldsymbol{F}\|_{s,G} + \|K\|_{s+1,G}), \qquad (4.3.4)$$

for an arbitrary disk $G_0 \Subset G$.

Proof. The inequality

$$\|\mathbf{\Phi}\|_{2,G} + \|P\|_{1,G} + t\|P\|_{2,G} \le C\big(\,\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}\,\big)$$

is proved in [3] for $K = 0$ when $a(\mathbf{\Psi}, \mathbf{\Phi})$ is simplified into $(\mathbf{grad}\,\mathbf{\Psi}, \mathbf{grad}\,\mathbf{\Phi})$, i.e., $(\mathcal{CE}(\mathbf{\Psi}), \mathcal{E}(\mathbf{\Phi}))$ is replaced by $(\mathbf{grad}\,\mathbf{\Psi}, \mathbf{grad}\,\mathbf{\Phi})$ in (4.3.1). By checking the proof there and using the fact that bilinear form $\big(\,\mathcal{CE}(\mathbf{\Psi}), \mathcal{E}(\mathbf{\Phi})\,\big)$ is coercive on space $\mathring{\boldsymbol{H}}^1$, we can conclude that the same estimate still applies to the current case. What we will do next is to follow the same proof to show that the estimate is still true for $K \ne 0$. At the same time, we will prove that $t^2\|P\|_{3,G}$ is also bounded above by the right hand side of (4.3.3).

Define $(\mathbf{\Phi}^0, P^0) \in \mathring{\boldsymbol{H}}^1(G) \times \hat{\boldsymbol{L}}^2(G)$ as the solution of (4.3.1) and (4.3.2) with t set equal to zero:

$$(\mathcal{CE}(\mathbf{\Psi}), \mathcal{E}(\mathbf{\Phi}^0)) - (P^0, \operatorname{rot}\mathbf{\Psi}) = (\mathbf{\Psi}, \boldsymbol{F}) \quad \text{for all } \mathbf{\Psi} \in \mathring{\boldsymbol{H}}^1(G), \tag{4.3.5}$$

$$-(\operatorname{rot}\mathbf{\Phi}^0, Q) = (Q, K) \quad \text{for all } Q \in L^2(G). \tag{4.3.6}$$

This is a Stokes like system which admits a unique solution. Moreover, the standard regularity theory gives [40]

$$\|\mathbf{\Phi}^0\|_{2,G} + \|P^0\|_{1,G} \le C\big(\,\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}\,\big). \tag{4.3.7}$$

From (4.3.1), (4.3.2), (4.3.5), and (4.3.6), we get

$$\big(\,\mathcal{CE}(\mathbf{\Phi} - \mathbf{\Phi}^0), \mathcal{E}(\mathbf{\Psi})\,\big) - \big(\,\mathbf{curl}(P - P^0), \mathbf{\Psi}\,\big) = 0 \quad \text{for all } \mathbf{\Psi} \in \mathring{\boldsymbol{H}}^1(G),$$

$$\big(\,\mathbf{\Phi} - \mathbf{\Phi}^0, \mathbf{curl}\,Q\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\,P, \mathbf{curl}\,Q\,\big) = 0 \quad \text{for all } Q \in H^1(G),$$

which imply

$$\begin{aligned} &\big(\,\mathcal{CE}(\mathbf{\Phi} - \mathbf{\Phi}^0), \mathcal{E}(\mathbf{\Psi})\,\big) - \big(\,\mathbf{curl}(P - P^0), \mathbf{\Psi}\,\big) + \big(\,\mathbf{\Phi} - \mathbf{\Phi}^0, \mathbf{curl}\,Q\,\big) \\ &\qquad + \lambda^{-1}t^2\big(\,\mathbf{curl}(P - P^0), \mathbf{curl}\,Q\,\big) \\ &= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,P^0, \mathbf{curl}\,Q\,\big) \quad \text{for all } (\mathbf{\Psi}, Q) \in \mathring{\boldsymbol{H}}^1(G) \times H^1(G). \end{aligned}$$

Choosing $\boldsymbol{\Psi} = \boldsymbol{\Phi} - \boldsymbol{\Phi}^0$ and $Q = P - P^0$, we obtain

$$\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G}^2 + t^2\|P - P^0\|_{1,G}^2 \leq Ct^2\|P^0\|_{1,G}\|P - P^0\|_{1,G}.$$

It easily follows that

$$\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G} + t\|P - P^0\|_{1,G} \leq Ct\|P^0\|_{1,G} \leq Ct(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}). \tag{4.3.8}$$

Hence also

$$\|P\|_{1,G} \leq C(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}).$$

Applying standard estimates for second-order elliptic problems to (4.3.1), we further obtain

$$\|\boldsymbol{\Phi}\|_{2,G} \leq C(\|P\|_{1,G} + \|\boldsymbol{F}\|_{0,G}) \leq C(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}). \tag{4.3.9}$$

Now from (4.3.2) and the definition of $\boldsymbol{\Phi}^0$ (i.e., (4.3.6)) we get

$$\lambda^{-1}t^2(\mathbf{curl}\, P, \mathbf{curl}\, Q) = -(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - (K, Q) = (\boldsymbol{\Phi}^0 - \boldsymbol{\Phi}, \mathbf{curl}\, Q) \quad \text{for all } Q \in H^1(G).$$

Thus P is the weak solution of the boundary value problem

$$-\Delta P = \lambda t^{-2}\operatorname{rot}(\boldsymbol{\Phi}^0 - \boldsymbol{\Phi}) \quad \text{in } G, \qquad \frac{\partial P}{\partial n} = 0 \quad \text{on } \partial G,$$

and by standard a priori estimates

$$\|P\|_{2,G} \leq Ct^{-2}\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{1,G} \leq Ct^{-1}(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,}) \tag{4.3.10}$$

and

$$\|P\|_{3,G} \leq Ct^{-2}\|\boldsymbol{\Phi} - \boldsymbol{\Phi}^0\|_{2,G} \leq Ct^{-2}(\|\boldsymbol{\Phi}\|_{2,G} + \|\boldsymbol{\Phi}^0\|_{2,G}) \leq Ct^{-2}(\|\boldsymbol{F}\|_{0,G} + \|K\|_{1,G}), \tag{4.3.11}$$

where we apply (4.3.8) in deriving (4.3.10), and (4.3.7), (4.3.9) in deriving (4.3.11). This completes the proof of (4.3.3).

In order to prove (4.3.4), we take a disk G_1 such that $G_0 \Subset G_1 \Subset G$. Find a cut-off function $\omega \in C_0^\infty(G_1)$ with $\omega = 1$ on G_0. We will use the notation $' = D_x$ or D_y, say for example, P' can be either P_x or P_y. Then, by differentiation rules, it is easy to obtain

$$\begin{aligned} -\,\mathbf{div}\, \mathcal{C}\mathcal{E}(\omega\boldsymbol{\Phi}') - \mathbf{curl}(\omega P') &= \omega \boldsymbol{F}' - \boldsymbol{J}(\omega, \boldsymbol{\Phi}') - P'\,\mathbf{curl}\,\omega \\ &=: \boldsymbol{F}_1, \end{aligned} \tag{4.3.12}$$

$$\begin{aligned} -\operatorname{rot}(\omega\boldsymbol{\Phi}') + \lambda^{-1}t^2\,\Delta(\omega P') &= \omega K' - \mathbf{curl}\,\omega \cdot \boldsymbol{\Phi}' + \lambda^{-1}t^2\,\Delta\,\omega P' \\ &\quad + 2\lambda^{-1}t^2\,\mathbf{grad}\,\omega \cdot \mathbf{grad}\,P' \\ &=: K_1, \end{aligned} \tag{4.3.13}$$

where

$$\boldsymbol{J}(\omega, \boldsymbol{\Phi}') =: \mathbf{div}\, \mathcal{C}\mathcal{E}(\omega\boldsymbol{\Phi}') - \omega\,\mathbf{div}\, \mathcal{C}\mathcal{E}(\boldsymbol{\Phi}'),$$

with

$$|\boldsymbol{J}(\omega, \boldsymbol{\Phi}')| \le C\|\boldsymbol{\Phi}'\|_{1,G_1}.$$

Obviously,

$$\begin{aligned} \int_{G_1} K_1 &= \int_{G_1} \big(-\operatorname{rot}(\omega\boldsymbol{\Phi}') + \lambda^{-1}t^2\,\Delta(\omega P') \big) \\ &= -\int_{\partial G_1} \big(\omega\boldsymbol{\Phi}' \cdot \boldsymbol{s} - \lambda^{-1}t^2 \partial(\omega P')/\partial n \big) = 0, \end{aligned}$$

because both ω and $\mathbf{grad}\,\omega$ vanish on ∂G_1. Moreover, we see that $(\omega\boldsymbol{\Phi}', \omega P')$ satisfies

$$(\mathcal{C}\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\omega\boldsymbol{\Phi}')) - (\mathbf{curl}(\omega P'), \boldsymbol{\Psi}) = (\boldsymbol{\Psi}, \boldsymbol{F}_1) \quad \text{for all } \boldsymbol{\Psi} \in \mathring{\boldsymbol{H}}^1(G_1), \tag{4.3.14}$$

$$-(\omega\boldsymbol{\Phi}', \mathbf{curl}\,Q) - \lambda^{-1}t^2(\mathbf{curl}\,Q, \mathbf{curl}(\omega P')) = (Q, K_1) \quad \text{for all } Q \in H^1(G_1). \tag{4.3.15}$$

Thus, (4.3.3) with $(\mathbf{\Phi}, P)$ replaced by $(\omega\mathbf{\Phi}', \omega P')$, G replaced by G_1, implies (for $\delta_1 = \int_{G_1} \omega P'$)

$$\|\omega\mathbf{\Phi}'\|_{2,G_1} + \|\omega P' - \delta_1\|_{1,G_1} + t\|\omega P' - \delta_1\|_{2,G_1} + t^2\|\omega P' - \delta_1\|_{3,G_1}$$
$$\leq C\big(\, \|\omega \boldsymbol{F}' - \boldsymbol{J}(\omega, \mathbf{\Phi}') - P' \,\mathbf{curl}\, \omega\|_{0,G_1}$$
$$+ \|\omega K' - \mathbf{curl}\, \omega \cdot \mathbf{\Phi}' + \lambda^{-1}t^2\, \Delta\, \omega P' + 2\lambda^{-1}t^2\, \mathbf{grad}\, \omega \cdot \mathbf{grad}\, P'\|_{1,G_1}\,\big)$$
$$\leq C\big(\, \|\boldsymbol{F}\|_{1,G} + \|K\|_{2,G}\,\big).$$

Since function $\omega = 1$ on G_0, inequality (4.3.4) is proved for $s = 1$. Now we are going to prove (4.3.4) for $s = 2$.

The notation $''$ now means either D_{xx}, or D_{xy}, or D_{yy}. Applying differentiation rules, we can obtain (for the same ω as in before)

$$-\,\mathbf{div}\; \mathcal{C}\mathcal{E}(\omega\mathbf{\Phi}'') - \mathbf{curl}(\omega P'') = \omega \boldsymbol{F}'' - \boldsymbol{J}(\omega, \mathbf{\Phi}'') - P''\,\mathbf{curl}\, \omega$$
$$=: \boldsymbol{F}_2, \tag{4.3.16}$$
$$-\,\mathrm{rot}(\omega\mathbf{\Phi}'') + \lambda^{-1}t^2\, \Delta(\omega P'') = \omega K'' - \mathbf{curl}\, \omega \cdot \mathbf{\Phi}''$$
$$+ \lambda^{-1}t^2\, \Delta\, \omega P'' + 2\lambda^{-1}t^2\, \mathbf{grad}\, \omega \cdot \mathbf{grad}\, P''$$
$$=: K_2, \tag{4.3.17}$$

with $\int_{G_1} K_2 = 0$. Then, inequality (4.3.3), with $(\mathbf{\Phi}, P)$ replaced by $(\omega\mathbf{\Phi}'', \omega P'')$ and G replaced by G_1, implies (for $\delta_2 = \int_{G_1} \omega P''$)

$$\|\omega\mathbf{\Phi}''\|_{2,G_1} + \|\omega P'' - \delta_2\|_{1,G_1} + t\|\omega P'' - \delta_2\|_{2,G_1} + t^2\|\omega P'' - \delta_2\|_{3,G_1}$$
$$\leq C\big(\, \|\omega \boldsymbol{F}'' - \boldsymbol{J}(\omega, \mathbf{\Phi}'') - P'' \,\mathbf{curl}\, \omega\|_{0,G_1}$$
$$+ \|\omega K'' - \mathbf{curl}\, \omega \cdot \mathbf{\Phi}'' + \lambda^{-1}t^2\, \Delta\, \omega P'' + 2\lambda^{-1}t^2\, \mathbf{grad}\, \omega \cdot \mathbf{grad}\, P''\|_{1,G_1}\,\big)$$
$$\leq \big(\, \|\boldsymbol{F}\|_{2,G_1} + \|K\|_{3,G_1} + \|\mathbf{\Phi}\|_{3,G_1} + \|P\|_{2,G_1} + t\|P\|_{3,G_1} + t^2\|P\|_{4,G_1}\,\big)$$
$$\leq C\big(\, \|\boldsymbol{F}\|_{2,G} + \|K\|_{3,G}\,\big),$$

where in the last step we use (4.3.4) with $s = 1$ and G_0 replaced by G_1. Since $\omega = 1$ on G_0, so inequality (4.3.4) is proved for $s = 2$. Same arguments, together with an induction on s could be used to prove (4.3.4) for $s \geq 3$. $\square$

4.4. The Arnold-Falk Element

Let $\mathcal{T}_h$ denote a family of quasi-uniform triangulations of Ω and $P_k(T)$ the set of polynomials of degree not greater than $k \geq 0$ restricted to T, an arbitrary element of $\mathcal{T}_h$. Consider the following finite element spaces

$$\begin{aligned}
Q_h &= \{q \in L^2 : q|_T \in P_0(T), \quad \text{for all } T \in \mathcal{T}_h\},\\
P_h &= \{p \in H^1 : p|_T \in P_1(T), \quad \text{for all } T \in \mathcal{T}_h\},\\
\hat{P}_h &= P_h \cap \hat{L}^2,\\
W_h &= \{w \in L^2 : w|_T \in P_1(T), \quad \text{for all } T \in \mathcal{T}_h, \text{ and } w \text{ is continuous at midpoints}\\
&\qquad \text{of element edges}\},\\
\bar{W}_h &= \{w \in L^2 : w|_T \in P_1(T), \quad \text{for all } T \in \mathcal{T}_h, \text{ and } w \text{ is continuous at midpoints}\\
&\qquad \text{of element edges and vanishes at midpoints of boundary edges}\}\\
\boldsymbol{V}_h &= \{\boldsymbol{\psi} \in \boldsymbol{H}^1 : \boldsymbol{\psi}|_T \in [P_1(T) \oplus B^3(T)]^2, \quad \text{for all } T \in \mathcal{T}_h\}.
\end{aligned}$$

In the above, $B^3(T)$ is the cubic bubble function on T. For $\Omega_0 \subseteq \Omega$, let

$$\begin{aligned}
\boldsymbol{V}_h(\Omega_0) &= \{\boldsymbol{\phi}|_{\Omega_0} \mid \boldsymbol{\phi} \in \boldsymbol{V}_h\}, & \mathring{\boldsymbol{V}}_h(\Omega_0) &= \{\boldsymbol{\phi} \in \boldsymbol{V}_h \mid \operatorname{supp} \boldsymbol{\phi} \subseteq \bar{\Omega}_0\},\\
P_h(\Omega_0) &= \{q|_{\Omega_0} \mid q \in P_h\}, & \mathring{P}_h(\Omega_0) &= \{q \in P_h \mid \operatorname{supp} q \subseteq \bar{\Omega}_0\},\\
W_h(\Omega_0) &= \{p|_{\Omega_0} \mid p \in W_h\}, & \mathring{W}_h(\Omega_0) &= \{p \in W_h \mid \operatorname{supp} p \subseteq \bar{\Omega}_0\}.
\end{aligned}$$

Since $\mathcal{T}_h$ is quasi-uniform, so the *approximation property*, *superapproximation property*, and the *inverse inequality property* introduced in section 3.3 hold for the above finite element spaces. We will not repeat them here.

Let P_h^0 be the local L^2-projection operator onto Q_h. Then the finite element in the primitive variables of Arnold-Falk (for the soft simply supported plate) reads as follows:

Find $(w_h, \boldsymbol{\phi}_h) \in \bar{W}_h \times \boldsymbol{V}_h$, such that

$$\big(C\,\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi}) \big) + \lambda t^{-2} \big(P_h^0 \boldsymbol{\phi}_h - \mathbf{grad}_h\, w_h, \boldsymbol{\psi} - \mathbf{grad}_h\, \mu \big) = \big(g, \mu \big), \qquad (4.4.1)$$

for all $(\mu, \boldsymbol{\psi}) \in \bar{W}_h \times \boldsymbol{V}_h$. Then, under the discrete Helmhotz theorem of Arnold and Falk [3]

$$[Q_h]^2 = \mathbf{grad}_h\, \bar{W}_h \oplus \mathbf{curl}\, P_h, \qquad (4.4.2)$$

the discrete shear vector can be expressed as

$$\zeta = \lambda t^{-2} (\mathbf{grad}_h\, w_h - P_h^0 \boldsymbol{\phi}_h) = \mathbf{grad}_h\, r_h + \mathbf{curl}\, p_h, \quad (r_h, p_h) \in \bar{W}_h \times \hat{P}_h. \quad (4.4.3)$$

Thus, equation (4.4.1) can be written equivalently in the form:

Find $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in \bar{W}_h \times \boldsymbol{V}_h \times \hat{P}_h \times \bar{W}_h$ such that

$$(\mathbf{grad}_h\, r_h, \mathbf{grad}_h\, \mu) = (g, \mu) \quad \text{for all } \mu \in \bar{W}_h, \qquad (4.4.4)$$

$$(C\,\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_h, \boldsymbol{\psi}) = (\mathbf{grad}_h\, r_h, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{V}_h, \qquad (4.4.5)$$

$$-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \hat{P}_h, \qquad (4.4.6)$$

$$(\mathbf{grad}_h\, w_h, \mathbf{grad}_h\, s) = (\boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}_h\, r_h, \mathbf{grad}_h\, s,) \quad \text{for all } s \in \bar{W}_h. \qquad (4.4.7)$$

The function r_h is uniquely determined by (4.4.4). Since the MINI element is stable [2], i.e., there is a constant C, such that

$$\sup_{\substack{\boldsymbol{\psi} \in \boldsymbol{V}_h \\ \boldsymbol{\psi} \neq 0}} \frac{(\mathbf{curl}\, p, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_1} \geq \sup_{\substack{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h \\ \boldsymbol{\psi} \neq 0}} \frac{(\mathbf{curl}\, p, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_1} \geq C \|p\|_0,$$

for all $p \in \hat{P}_h$, so $(\boldsymbol{\phi}_h, p_h)$ is uniquely defined by equations (4.4.5) and (4.4.6), and thereafter, w_h by equation (4.4.7).

It is important to note that system (4.4.4)-(4.4.7) is for the purpose of convergence analysis only. Equation (4.4.1) is the one used for the actual computation. Our interior analysis of the Arnold-Falk element will also be based on the decoupled system of Poisson's equations and the Stokes-like equations, not the original Mindlin-Reissner plate system (i.e., (4.2.1)). Therefore, the interior estimate of the Arnold-Falk element consists of obtaining the interior estimate for the nonconforming element for the Poisson equation and that for the MINI element for the perturbed system (2.3.4) and (2.3.4). Since the first part is done in Chapter 2, we need only concentrate on the Stokes-like system here.

Before we turn to the next section, we introduce a result on the convergence of the MINI element for the perturbed Stokes-like system.

Lemma 4.4.1. *Let G_h a union of triangles. Then for $\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(G_h)$, $p \in H^1(G_h)$, and $\boldsymbol{F} \in L^2(G_h)$, there exist unique functions $\pi\boldsymbol{\phi} \in \mathring{\boldsymbol{V}}_h(G_h)$ and $\pi p \in P_h(G_h)$ with $\int_{G_h} p = \int_{G_h} \pi p$, such that*

$$\begin{aligned} &\big(\, C\mathcal{E}(\boldsymbol{\phi} - \pi\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,\mathbf{curl}(p - \pi p), \boldsymbol{\psi}\,\big) = (\boldsymbol{F}, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h), \\ -&\big(\,\boldsymbol{\phi} - \pi\boldsymbol{\phi}, \mathbf{curl}\, q\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}(p - \pi p), \mathbf{curl}\, q\,\big) = 0 \qquad \text{for all } q \in P_h(G_h). \end{aligned}$$

Moreover,

$$\begin{aligned} &\|\boldsymbol{\phi} - \pi\boldsymbol{\phi}\|_{1,G_h} + \|p - \pi p\|_{0,G_h} + t\|\,\mathbf{curl}(p - \pi p)\|_{0,G_h} \\ \le\, & C\big(\inf_{q \in P_h(G_h)} (\|p - q\|_{0,G_h} + t\|\,\mathbf{curl}(p - q)\|_{0,G_h}) \\ &+ \inf_{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h)} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,G_h} + \|\boldsymbol{F}\|_{0,G_h}\,\big). \end{aligned} \tag{4.4.8}$$

Proof. The unique existence of solution $(\pi\boldsymbol{\phi}, \pi p)$ follows from the stability property of the MINI element and Brezzi's Theorem [13] . The estimate (4.4.8) can be obtained by following the proof in [3, Theorem 5.5]. $\square$

4.5 Interior Duality Estimates

Let $(w, \boldsymbol{\phi}) \in H^1 \times \boldsymbol{H}^1$ be some solution to the Reissner-Mindlin plate equations and $(r, p) \in H^1 \times H^1$ be determined by the Helmholtz decomposition (4.2.2). Regardless of the boundary conditions used to specify the particular solution, $(r, \boldsymbol{\phi}, p, w)$ satisfies

$$
\begin{aligned}
(\mathbf{grad}\, r, \mathbf{grad}\, \mu) &= (g, \mu) \quad \text{for all } \mu \in \mathring{H}^1, \\
\big(C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}\, p, \boldsymbol{\psi} \big) &= \big(\mathbf{grad}\, r, \boldsymbol{\psi} \big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1, \\
-(\boldsymbol{\phi}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p, \mathbf{curl}\, q) &= 0 \quad \text{for all } q \in \mathring{H}^1, \\
(\mathbf{grad}\, w, \mathbf{grad}\, s) &= (\boldsymbol{\phi} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s) \quad \text{for all } s \in \mathring{H}^1.
\end{aligned}
$$

Similarly, regardless of the particular boundary conditions, the finite element solutions $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in W_h \times \boldsymbol{V}_h \times P_h \times W_h$ satisfies

$$
\begin{aligned}
(\mathbf{grad}\, r_h, \mathbf{grad}\, \mu) &= (g, \mu) \quad \text{for all } \mu \in \mathring{W}_h, \\
\big(C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathbf{curl}\, p_h, \boldsymbol{\psi} \big) &= \big(\mathbf{grad}_h\, r_h, \boldsymbol{\psi} \big) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \\
-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p_h, \mathbf{curl}\, q) &= 0 \quad \text{for all } q \in \mathring{P}_h, \\
(\mathbf{grad}\, w_h, \mathbf{grad}_h\, s) &= (\boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}\, r_h, \mathbf{grad}_h\, s) \quad \text{for all } s \in \mathring{W}_h.
\end{aligned}
$$

Then, together with integration by parts, we obtain

$$(\mathbf{grad}_h(r - r_h), \mathbf{grad}_h \mu) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n} v \quad \text{for all } \mu \in \mathring{W}_h, \tag{4.5.1}$$

$$(C\,\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}(p - p_h), \boldsymbol{\psi}) = (\mathbf{grad}_h(r - r_h), \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.5.2}$$

$$-(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}(p - p_h), \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h, \tag{4.5.3}$$

$$(\mathbf{grad}_h(w - w_h), \mathbf{grad}_h s) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}_h(r - r_h), \mathbf{grad}_h s) - \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\phi} \cdot \boldsymbol{n}_T + \lambda^{-1} t^2 \frac{\partial r}{\partial n}) s \quad \text{for all } s \in \mathring{W}_h. \tag{4.5.4}$$

As usual, the interior error analysis starts from these interior variational discretization equations. They are independent of the boundary conditions.

The interior estimate for $r - r_h$ is done in Chapter 2 (Theorem 2.5.2). However, we cannot use Theorem 2.5.2 directly to obtain the interior estimate for $w - w_h$. This is caused by the difference between (2.4.1) and (4.5.4). But we can still use the same idea (as in the proof of Theorem 2.5.2) to get the interior estimate for $(w - w_h)$. This will be done in Theorem 4.6.3.

So it is only necessary to obtain interior estimates for $(\boldsymbol{\phi} - \boldsymbol{\phi}_h, p - p_h)$ which satisfy (4.5.2) and (4.5.3). These will be the focus of this and next sections. What we will do is to use the same two-step approach (the interior duality estimate and the interior error estimate, as for the Stokes equations in Chapter 3) to obtain the interior estimate for the MINI element.

Consider functions $\boldsymbol{\phi} \in \boldsymbol{H}^1$ and $p \in H^1$ that satisfy the variational equations

$$(C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p, \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.5.5}$$

$$-(\boldsymbol{\phi}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, q, \mathbf{curl}\, p) = 0 \quad \text{for all } q \in \mathring{P}_h. \tag{4.5.6}$$

We have the following result.

Theorem 4.5.1. *Assume* $\boldsymbol{\phi} \in \boldsymbol{H}^1$ *and* $p \in H^1$ *satisfy* (4.5.5) *and* (4.5.6). *Let* $G_0 \Subset G \Subset \Omega$ *be two concentric disks. Then for any integer* $\alpha \geq 0$, *the following holds*

$$\|\boldsymbol{\phi}\|_{0,G_0}+\|p\|_{-1,G_0} \leq C(h\|\boldsymbol{\phi}\|_{1,G}+h\|p\|_{0,G}+ht\|\,\mathbf{curl}\,p\|_{0,G}+\|\boldsymbol{\phi}\|_{-\alpha,G}+\|p\|_{-\alpha-1,G}). \tag{4.5.7}$$

Proof. Find a disk G_1 such that $G_0 \Subset G_1 \Subset G$ and construct a function $\omega \in C_0^\infty(G_1)$ with $\omega = 1$ on G_0. Then, for any non-negative integer s,

$$\|\boldsymbol{\phi}\|_{-s,G_0} \leq \|\omega\boldsymbol{\phi}\|_{-s,G} = \sup_{\substack{\boldsymbol{F}\in\mathring{\boldsymbol{H}}^s(G)\\ \boldsymbol{F}\neq 0}} \frac{(\omega\boldsymbol{\phi},\boldsymbol{F})}{\|\boldsymbol{F}\|_{s,G}}. \tag{4.5.8}$$

To estimate the right hand side of (4.5.8), we define $(\boldsymbol{\Phi}, P)$ through (4.3.1) and (4.3.2) with $K = 0$. Then take $\boldsymbol{\Psi} = \omega\boldsymbol{\phi}$ in (4.3.1) to obtain

$$\begin{aligned}
\big(\,\omega\boldsymbol{\phi},\boldsymbol{F}\,\big) &= \big(\,C\mathcal{E}(\omega\boldsymbol{\phi}),\mathcal{E}(\boldsymbol{\Phi})\,\big) - \big(\,\omega\boldsymbol{\phi},\mathbf{curl}\,P\,\big)\\
&= \big(\,C\mathcal{E}(\boldsymbol{\phi}),\mathcal{E}(\omega\boldsymbol{\Phi})\,\big) - \big(\,\boldsymbol{\phi},\mathbf{curl}(\omega P)\,\big) - \mathrm{R}(\omega,\boldsymbol{\Phi},\boldsymbol{\phi}) + \big(\,\mathbf{curl}\,\omega, P\boldsymbol{\phi}\,\big)\\
&= \big\{\big(\,C\mathcal{E}(\boldsymbol{\phi}),\mathcal{E}(\omega\boldsymbol{\Phi})^I\,\big) - \big(\,\boldsymbol{\phi},\mathbf{curl}(\omega P)\,\big)\big\}\\
&\quad + \big\{\big(\,C\mathcal{E}(\boldsymbol{\phi}),\mathcal{E}[\omega\boldsymbol{\Phi}-(\omega\boldsymbol{\Phi})^I]\,\big) - \mathrm{R}(\omega,\boldsymbol{\Phi},\boldsymbol{\phi}) + \big(\,\mathbf{curl}\,\omega, P\boldsymbol{\phi}\,\big)\big\}\\
&=: A_1 + B_1.
\end{aligned} \tag{4.5.9}$$

Here the superscript I is the approximation operator. Chosing $\boldsymbol{\psi}$ to be $(\omega\boldsymbol{\phi})^I$ in

(4.5.5) we get

$$\begin{aligned} A_1 &= \big(\, \mathbf{curl}\, p, (\omega \mathbf{\Phi})^I \,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) \\ &= \big(\, \mathbf{curl}\, p, \omega \mathbf{\Phi} \,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) + \big(\, \mathbf{curl}\, p, (\omega \mathbf{\Phi})^I - \omega \mathbf{\Phi} \,\big) \\ &= \big\{ \big(\, \mathbf{curl}(\omega p), \mathbf{\Phi} \,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) \big\} \\ &\quad - \big\{ \big(\, \mathbf{curl}\, \omega, p \mathbf{\Phi} \,\big) - \big(\, \mathbf{curl}\, p, (\omega \mathbf{\Phi})^I - \omega \mathbf{\Phi} \,\big) \big\} \\ &=: A_2 + B_2. \end{aligned} \tag{4.5.10}$$

Taking $Q = \omega p$ in (4.3.2) (with $K = 0$), we obtain

$$\begin{aligned} A_2 &= -\lambda^{-1} t^2 \big(\, \mathbf{curl}(\omega p), \mathbf{curl}\, P \,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) \\ &= -\lambda^{-1} t^2 \big(\, \mathbf{curl}\, p, \mathbf{curl}(\omega P)\,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) + \lambda^{-1} t^2 \mathrm{R}'(\omega, P, p) \\ &= -\, \big\{ \lambda^{-1} t^2 \big(\, \mathbf{curl}\, p, \mathbf{curl}(\omega P)^I \,\big) + \big(\, \phi, \mathbf{curl}(\omega P)\,\big) \big\} \\ &\quad + \big\{ \lambda^{-1} t^2 \big(\, \mathbf{curl}\, p, \mathbf{curl}[(\omega P)^I - \omega P] \,\big) + \lambda^{-1} t^2 \mathrm{R}'(\omega, P, p) \big\} =: A_3 + B_3. \end{aligned} \tag{4.5.11}$$

Substituting $q = (\omega P)^I$ in (4.5.6), we have

$$A_3 = \big(\, \phi, \mathbf{curl}(\omega P)^I \,\big) - \big(\, \phi, \mathbf{curl}(\omega P)\,\big) = \big(\, \phi, \mathbf{curl}[(\omega P)^I - \omega P] \,\big). \tag{4.5.12}$$

Combining (4.5.9) through (4.5.12), we get

$$\big(\, \omega \phi, \boldsymbol{F} \,\big) = B_1 + B_2 + B_3 + A_3. \tag{4.5.13}$$

Then applying the approximation property, (4.2.7), (4.2.8), integration by parts, and the Schwarz inequality, we obtain

$$\begin{aligned} \mid B_1 \mid &\le C \big(\, h \|\phi\|_{1,G_1} \|\mathbf{\Phi}\|_{2,G_1} + \|\phi\|_{-s-1,G_1} (\|\mathbf{\Phi}\|_{s+2,G_1} + \|P\|_{s+1,G_1}) \,\big), \\ \mid B_2 \mid &\le C \big(\, h \|p\|_{0,G_1} \|\mathbf{\Phi}\|_{2,G_1} + \|p\|_{-s-2,G_1} \|\mathbf{\Phi}\|_{s+2,G_1} \,\big), \\ \mid B_3 \mid &\le C \big(\, h t^2 \|\, \mathbf{curl}\, p\|_{0,G_1} \|P\|_{2,G_1} + t^2 \|p\|_{-s-2,G_1} \|P\|_{s+3,G_1} \,\big), \\ \mid A_3 \mid &\le C h \|\phi\|_{1,G_1} \|P\|_{1,G_1}. \end{aligned} \tag{4.5.14}$$

First combining (4.5.14), (4.5.13), and (4.5.8), then applying (4.3.3) and (4.3.4), we obtain

$$\|\phi\|_{-s,G_0} \le C\big(h\|\phi\|_{1,G} + h\|p\|_{0,G} + \|\phi\|_{-s-1,G} + \|p\|_{-s-2,G} + ht\|\,\mathbf{curl}\,p\|_{0,G}\,\big). \tag{4.5.15}$$

To estimate $\|p\|_{-s-1,G_0}$, first find a function $\delta \in C_0^\infty(G_1)$ with $\int_G \delta = 1$. Then,

$$\|p\|_{-s-1,G_0} \le \|\omega p\|_{-s-1,G} = \sup_{\substack{g\in \mathring{H}^{s+1}(G)\\ g\neq 0}} \frac{(\omega p, g)}{\|g\|_{s+1,G}}. \tag{4.5.16}$$

Note that

$$(\omega p, g) = (\omega p, g - \delta \int_G g) + (\omega p, \delta \int_G g) \tag{4.5.17}$$

and

$$|\,(\omega p, \delta \int_G g)\,| \le C\|p\|_{-s-1,G}\cdot \|g\|_{0,G}. \tag{4.5.18}$$

In order to estimate the first term on the right hand side of (4.5.17), we define $(\mathbf{\Phi}, P)$ through (4.3.1) and (4.3.2) with $\boldsymbol{F} = 0$, $K = g - \delta\int_G g$. Taking $Q = \omega p$ in (4.3.2), we have

$$\begin{aligned}
&\big(\,\omega p, g - \delta \int_G g\,\big)\\
&= -\big(\,\mathbf{curl}(\omega p), \mathbf{\Phi}\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}(\omega p), \mathbf{curl}\,P\,\big)\\
&= -\big(\,\mathbf{curl}\,p, \omega\mathbf{\Phi}\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}(\omega P)\,\big) - \big(\,\mathbf{curl}\,\omega, p\mathbf{\Phi}\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\\
&= -\big\{\big(\,\mathbf{curl}\,p, (\omega\mathbf{\Phi})^I\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}(\omega P)^I\,\big)\big\}\\
&\quad + \big\{\big(\,\mathbf{curl}\,p, (\omega\mathbf{\Phi})^I - \omega\mathbf{\Phi}\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\,p, \mathbf{curl}[(\omega P)^I - \omega P]\,\big)\\
&\quad - \big(\,\mathbf{curl}\,\omega, p\mathbf{\Phi}\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, P, p)\big\}\\
&=: C_1 + D_1.
\end{aligned} \tag{4.5.19}$$

Applying (4.5.5) and (4.5.6) with $\boldsymbol{\psi} = (\omega\boldsymbol{\Phi})^I$ and $q = (\omega P)^I$, respectively, we get

$$\begin{aligned} C_1 &= -\big(\, C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\omega\boldsymbol{\Phi})^I \,\big) + \big(\, \boldsymbol{\phi}, \mathbf{curl}(\omega P)^I \,\big) \\ &= -\big(\, C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\omega\boldsymbol{\Phi}) \,\big) + \big(\, \boldsymbol{\phi}, \mathbf{curl}(\omega P)^I \,\big) + \big(\, C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[w\boldsymbol{\Phi} - (\omega\boldsymbol{\Phi})^I] \,\big) \\ &= -\big\{\big(\, C\mathcal{E}(\omega\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\Phi}) \,\big) - \big(\, \boldsymbol{\phi}, \mathbf{curl}(\omega P)^I \,\big)\big\} \\ &\quad + \big\{\big(\, C\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}[\omega\boldsymbol{\Phi} - (\omega\boldsymbol{\Phi})^I] \,\big) - \mathrm{R}(\omega, \boldsymbol{\Phi}, \boldsymbol{\phi})\big\} \\ &=: C_2 + D_2. \end{aligned} \tag{4.5.20}$$

Taking $\boldsymbol{\Psi} = \omega\boldsymbol{\phi}$ in (4.3.1) (with $\boldsymbol{F} = 0$), we obtain

$$C_2 = -\big(\, \omega\boldsymbol{\phi}, \mathbf{curl}\, P \,\big) + \big(\, \boldsymbol{\phi}, \mathbf{curl}(\omega P)^I \,\big) = \big(\, \boldsymbol{\phi}, \mathbf{curl}[(\omega P)^I - \omega P] \,\big) + \big(\, \mathbf{curl}\, \omega, P\boldsymbol{\phi} \,\big). \tag{4.5.21}$$

So far, we have

$$(\omega p, g - \delta \int g) = D_1 + D_2 + C_2.$$

Then applying (4.2.7), (4.2.8), integration by parts, the approximation property, and Schwarz inequality, we arrive at

$$\begin{aligned} |\, D_1 \,| &\le C\big(\, h\|p\|_{0,G_1}\|\boldsymbol{\Phi}\|_{2,G_1} + ht^2\|\,\mathbf{curl}\, p\|_{0,G}\|P\|_{2,G_1} \\ &\qquad + \|p\|_{-s-2,G}\|\boldsymbol{\Phi}\|_{s+2,G_1} + t^2\|p\|_{-s-2,G}\|P\|_{s+3,G_1} \,\big), \\ |\, D_2 \,| &\le C\big(\, h\|\boldsymbol{\phi}\|_{1,G}\|\boldsymbol{\Phi}\|_{2,G_1} + \|\boldsymbol{\phi}\|_{-s-1,G_1}\|\boldsymbol{\Phi}\|_{s+2,G_1} \,\big), \\ |\, C_2 \,| &\le C\big(\, h\|\boldsymbol{\phi}\|_{1,G_1}\|P\|_{1,G_1} + \|\boldsymbol{\phi}\|_{-s-1,G_1}\|P\|_{s+1,G_1} \,\big). \end{aligned} \tag{4.5.22}$$

Combining (4.5.16) through (4.5.22), together with (4.3.3) and (4.3.4), we obtain

$$\|p\|_{-s-1,G_0} \le C\big(\, h\|\boldsymbol{\phi}\|_{1,G} + h\|p\|_{0,G} + \|\boldsymbol{\phi}\|_{-s-1,G} + \|p\|_{-s-2,G} + ht\|\,\mathbf{curl}\, p\|_{0,G} \,\big). \tag{4.5.23}$$

Finally, (4.5.7) can be obtained by the standard iteration method (cf. section 2.4, section 3.4). $\square$

4.6 Interior Error Estimates

In this section we first obtain the interior estimate of the MINI element for the Stokes-like equations with perturbation, then we use it to derive the interior estimate for the Arnold-Falk element (Theorem 4.6.3). To be specific, Lemma 4.6.1 gives a bound on functions satisfying a homogeneous discrete Stokes-like equations. It is then used with Theorem 4.5.1 to get the interior estimate for the MINI element for the Stokes-like system (Theorem 4.6.2). By combining this result with the interior estimate of the nonconforming element (Theorem 2.5.2), we obtain the interior estimate of the Arnold-Falk element (Theorem 4.6.3). This is the main result of this chapter.

Lemma 4.6.1. *Suppose* $(\boldsymbol{\phi}_h, p_h) \in \boldsymbol{V}_h \times P_h$ *is such that*

$$(C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}\, p_h, \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.6.1}$$

$$-(\boldsymbol{\phi}_h, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p_h, \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h. \tag{4.6.2}$$

Then, for any two concentric disks $G_0 \Subset G \Subset \Omega$*,* h *small enough,* α *and* β *any nonnegative integers, we have*

$$\begin{aligned} &\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\, p_h\|_{0,G_0} \\ &\le C\big(t^\beta(\|\boldsymbol{\phi}_h\|_{1,G} + t\|p_h\|_{1,G}) + \|\boldsymbol{\phi}_h\|_{-\alpha,G} + \|p_h\|_{-\alpha-1,G}\big), \end{aligned} \tag{4.6.3}$$

where $C = C(\alpha, \beta, G_0, G)$*.*

Proof. Let $G_0 \Subset G' \Subset G_h \Subset G_1 \Subset G$ with G' a concentric disk and G_h a union of elements. Construct $\omega \in C_0^\infty(G')$ with $\omega \equiv 1$ on G_0. Set $\widetilde{\boldsymbol{\phi}_h} = \omega\boldsymbol{\phi}_h$, $\widetilde{p_h} = \omega p_h$. Then

$\widetilde{\boldsymbol{\phi}_h} \in \overset{\circ}{\boldsymbol{H}}{}^1(G_h)$, $\widetilde{p_h} \in H^1(G_h)$. By Lemma 4.4.1, $\pi\widetilde{\boldsymbol{\phi}_h} \in \overset{\circ}{\boldsymbol{V}}_h(G_h)$ and $\pi\widetilde{p_h} \in P_h(G_h)$ can be uniquely determined by the equations

$$\big(\, C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \boldsymbol{\psi}\,\big) = 0 \quad \text{for all} \quad \boldsymbol{\psi} \in \overset{\circ}{\boldsymbol{V}}_h(G_h), \tag{4.6.4}$$

$$-\big(\,\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}, \mathbf{curl}\, q\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, q\,\big) = 0 \quad \text{for all } q \in P_h(G_h), \tag{4.6.5}$$

with $\int_{G_h} \widetilde{p_h} = \int_{G_h} \pi\widetilde{p_h}$. Moreover, we have

$$\begin{aligned}
&\|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} + t\|\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h})\|_{0,G_h} \\
&\le C\big(\inf_{\boldsymbol{\psi}\in\overset{\circ}{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}_h} - \boldsymbol{\psi}\|_{1,G_h} + \inf_{q\in P_h(G_h)} (\|\widetilde{p_h} - q\|_{0,G_h} + t\|\,\mathbf{curl}(\widetilde{p_h} - q)\|_{0,G_h}\,)\big) \\
&\le Ch\big(\,\|\boldsymbol{\phi}_h\|_{1,G_h} + \|p_h\|_{0,G_h} + t\|p_h\|_{1,G_h}\,\big),
\end{aligned} \tag{4.6.6}$$

where we have used the superapproximation property (cf. section 3.3) in the last step. By the triangle inequality

$$\begin{aligned}
&\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\, p_h\|_{0,G_0} \\
&\le \|\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h}\|_{0,G_h} + t\|\,\mathbf{curl}\,\widetilde{p_h}\|_{0,G_h} \\
&\le \|\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\widetilde{p_h} - \pi\widetilde{p_h}\|_{0,G_h} + t\|\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h})\|_{0,G_h} \\
&\quad + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h} + t\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|_{0,G_h} \\
&\le Ch(\|\boldsymbol{\phi}_h\|_{1,G_h} + \|p_h\|_{0,G_h} + t\|p_h\|_{1,G_h}) \\
&\quad + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + \|\pi\widetilde{p_h}\|_{0,G_h} + t\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned} \tag{4.6.7}$$

We shall consider $\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}$ first. In (4.6.4), we take $\boldsymbol{\psi} = \pi\widetilde{\boldsymbol{\phi}_h}$ to obtain

$$\big(\, C\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) = \big(\, C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) - \big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \pi\widetilde{\boldsymbol{\phi}_h}\,\big). \tag{4.6.8}$$

We have

$$
\begin{aligned}
&\big(\, C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) = \big(\, C\mathcal{E}(\omega\boldsymbol{\phi}_h), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big)\\
&= \big(\, C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\omega\pi\widetilde{\boldsymbol{\phi}_h})\,\big) - \mathrm{R}(\omega, \pi\widetilde{\boldsymbol{\phi}_h}, \boldsymbol{\phi}) = \big(\, C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I]\,\big)\\
&\quad + \Big\{ \big(\, C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[\omega\pi\widetilde{\boldsymbol{\phi}_h} - (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I]\,\big) - \mathrm{R}(\omega, \pi\widetilde{\boldsymbol{\phi}_h}, \boldsymbol{\phi}) \Big\}\\
&=: \big(\, C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I]\,\big) + F_1.
\end{aligned}
\tag{4.6.9}
$$

Taking $\boldsymbol{\psi} = (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I$ in (4.6.1), we get

$$
\begin{aligned}
&\big(\, C\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[(\omega\pi\widetilde{\boldsymbol{\phi}_h})^I]\,\big) = \big(\, \mathbf{curl}\, p_h, (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I \,\big)\\
&= \big(\, \mathbf{curl}\, p_h, \omega\pi\widetilde{\boldsymbol{\phi}_h} \,\big) + \big(\, \mathbf{curl}\, p_h, (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I - \omega\pi\widetilde{\boldsymbol{\phi}_h} \,\big)\\
&= \big(\, \mathbf{curl}(\omega p_h), \pi\widetilde{\boldsymbol{\phi}_h} \,\big) - \Big\{ \big(\, \mathbf{curl}\, \omega, p_h\pi\widetilde{\boldsymbol{\phi}_h} \,\big) - \big(\, \mathbf{curl}\, p_h, (\omega\pi\widetilde{\boldsymbol{\phi}_h})^I - \omega\pi\widetilde{\boldsymbol{\phi}_h} \,\big) \Big\}\\
&=: \big(\, \mathbf{curl}\, \widetilde{p_h}, \pi\widetilde{\boldsymbol{\phi}_h} \,\big) + F_2.
\end{aligned}
\tag{4.6.10}
$$

Combining (4.6.8)–(4.6.10) and substituting $q = \pi\widetilde{p_h}$ in (4.6.5), we obtain

$$
\begin{aligned}
&\big(\, C\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h})\,\big) = \big(\, \mathbf{curl}\, \pi\widetilde{p_h}, \pi\widetilde{\boldsymbol{\phi}_h} \,\big) + F_1 + F_2\\
&= \big(\, \widetilde{\boldsymbol{\phi}_h}, \mathbf{curl}\, \pi\widetilde{p_h} \,\big) + \lambda^{-1}t^2 \big(\, \mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, \pi\widetilde{p_h} \,\big) + F_1 + F_2\\
&= \big(\, \omega\boldsymbol{\phi}_h, \mathbf{curl}\, \pi\widetilde{p_h} \,\big) + \lambda^{-1}t^2 \big(\, \mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, \pi\widetilde{p_h} \,\big) + F_1 + F_2\\
&= \big(\, \boldsymbol{\phi}_h, \mathbf{curl}(\omega\pi\widetilde{p_h})\,\big) - \big(\, \mathbf{curl}\, \omega, \pi\widetilde{p_h}\boldsymbol{\phi}_h \,\big) + \lambda^{-1}t^2 \big(\, \mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, \pi\widetilde{p_h} \,\big)\\
&\quad + F_1 + F_2\\
&= \big\{ \big(\, \boldsymbol{\phi}_h, \mathbf{curl}(\omega\pi\widetilde{p_h})^I \,\big) + \lambda^{-1}t^2 \big(\, \mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\, \pi\widetilde{p_h} \,\big) \big\}\\
&\quad + \big\{ \big(\, \boldsymbol{\phi}_h, \mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big) - \big(\, \mathbf{curl}\, \omega, \pi\widetilde{p_h}\boldsymbol{\phi}_h \,\big) \big\} + F_1 + F_2\\
&=: E_1 + F_3 + F_1 + F_2.
\end{aligned}
\tag{4.6.11}
$$

Setting $q = (\omega\pi\widetilde{p_h})^I$ in (4.6.2), we get

$$\begin{aligned}
E_1 &= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h, \mathbf{curl}(\omega\pi\widetilde{p_h})^I\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\,\pi\widetilde{p_h}\,\big) \\
&= -\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h, \mathbf{curl}(\omega\pi\widetilde{p_h})\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h, \mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big) \\
&\quad + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\,\pi\widetilde{p_h}\,\big) \\
&= -\lambda^{-1}t^2(\mathbf{curl}(\omega p_h), \mathbf{curl}\,\pi\widetilde{p_h}) + \lambda^{-1}t^2\big(\,\mathbf{curl}(\widetilde{p_h} - \pi\widetilde{p_h}), \mathbf{curl}\,\pi\widetilde{p_h}\,\big) \\
&\quad + \big\{\lambda^{-1}t^2\big(\,\mathbf{curl}\,p_h, \mathbf{curl}[\omega\pi\widetilde{p_h} - (\omega\pi\widetilde{p_h})^I]\,\big) + \lambda^{-1}t^2\mathrm{R}'(\omega, p_h, \pi\widetilde{p_h})\big\} \\
&=: -\lambda^{-1}t^2(\mathbf{curl}\,\pi\widetilde{p_h}, \mathbf{curl}\,\pi\widetilde{p_h}) + F_4. \qquad (4.6.12)
\end{aligned}$$

So far, we have

$$\|C^{\frac{1}{2}}\mathcal{E}(\pi\widetilde{\phi_h})\|^2_{0,G_h} + \lambda^{-1}t^2\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|^2_{0,G_h} = F_1 + F_2 + F_3 + F_4. \qquad (4.6.13)$$

Using the superapproximation properties (cf. section 3.3), the Schwarz inequality, integration by parts, (4.2.7), and (4.2.8), we obtain

$$\begin{aligned}
|\,F_1\,| &\le C\big(\,h\|\phi_h\|_{1,G_h}\|\pi\widetilde{\phi_h}\|_{1,G_h} + \|\phi_h\|_{0,G_h}\|\pi\widetilde{\phi_h}\|_{1,G_h}\,\big), \\
|\,F_2\,| &\le C\big(\,\|p_h\|_{-1,G_h}\|\pi\widetilde{\phi_h}\|_{1,G_h} + h\|p_h\|_{0,G_h}\|\pi\widetilde{\phi_h}\|_{1,G_h}\,\big), \\
|\,F_3\,| &\le C\big(\,h\|\phi_h\|_{1,G_h}\|\pi\widetilde{p_h}\|_{0,G_h} + \|\phi_h\|_{0,G_h}\|\pi\widetilde{p_h}\|_{0,G_h}\,\big), \\
|\,F_4\,| &\le Ct^2\big(\,h\|\,\mathbf{curl}\,p_h\|_{0,G_h}\|\pi\widetilde{p_h}\|_{1,G_h} + \|p_h\|_{1,G_h}\|\pi\widetilde{p_h}\|_{0,G_h}\,\big).
\end{aligned}$$

Combining the above inequalities with (4.6.13), using the inverse inequalities for $\pi\widetilde{p_h}$, ϕ_h, and p_h, we get

$$\begin{aligned}
&\|\pi\widetilde{\phi_h}\|^2_{1,G_h} + t^2\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|^2_{0,G_h} \\
&\le C\big(\,h\|\phi_h\|_{1,G_h} + \|\phi_h\|_{0,G_h} + h\|p_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\,\big)\|\pi\widetilde{\phi_h}\|_{1,G_h} \\
&\quad + C\big(\,h\|\phi_h\|_{1,G_h} + \|\phi_h\|_{0,G_h} + t^2\|p_h\|_{1,G_h}\,\big)\|\pi\widetilde{p_h}\|_{0,G_h} \\
&\le C\big(\,\|\phi_h\|_{0,G_h} + \|p_h\|_{-1,G_h}\,\big)\|\pi\widetilde{\phi_h}\|_{1,G_h} + C\big(\,\|\phi_h\|_{0,G_h} + t^2\|p_h\|_{1,G_h}\,\big)\|\pi\widetilde{p_h}\|_{0,G_h}.
\end{aligned}$$

(4.6.14)

To proceed, we need to estimate $\|\pi\widetilde{p_h}\|_{0,G_h}$. By the triangle inequality,

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le \|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\text{meas}(G_h)}\|_{0,G_h} + \frac{\|\int_{G_h}\pi\widetilde{p_h} - \widetilde{p_h}\|_{0,G_h}}{\text{meas}(G_h)} + \frac{\|\int_{G_h}\widetilde{p_h}\|_{0,G_h}}{\text{meas}(G_h)}. \tag{4.6.15}$$

It is easy to see that the second term on the right hand side of the above inequality is bounded by the right hand side of (4.6.6). For the last term we have

$$\|\int_{G_h}\widetilde{p_h}\|_{0,G_h} = \|\int_{G_h}\omega p_h\|_{0,G_h} \le C\|p_h\|_{-1,G_h}.$$

From that fact that the triangulation $\mathcal{T}_h$ is quasi-uniform, we have the following stability condition for the MINI element on set G_h,

$$\|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\text{meas}(G_h)}\|_{0,G_h} \le C \sup_{\substack{\boldsymbol{\psi}\in\hat{\boldsymbol{V}}_h(G_h)\\ \boldsymbol{\psi}\neq 0}} \frac{(\,\mathbf{curl}\,\pi\widetilde{p_h}, \boldsymbol{\psi}\,)_{G_h}}{\|\boldsymbol{\psi}\|_{1,G_h}}. \tag{4.6.16}$$

Applying (4.6.4), we obtain

$$\begin{aligned}(\mathbf{curl}\,\pi\widetilde{p_h}, \boldsymbol{\psi}) &= (\,\mathbf{curl}\,\widetilde{p_h}, \boldsymbol{\psi}\,) - (\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\boldsymbol{\psi})\,)\\ &= (\,\mathbf{curl}(\omega p_h), \boldsymbol{\psi}\,) - (\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})\,)\\ &= (\,\mathbf{curl}\,p_h, w\boldsymbol{\psi}\,) - (\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})\,) + (\,\mathbf{curl}\,\omega, p_h\boldsymbol{\psi}\,)\\ &= \Big\{(\,\mathbf{curl}\,p_h, (\omega\boldsymbol{\psi})^I\,) - (\,C\mathcal{E}(\widetilde{\boldsymbol{\phi}_h} - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})\,)\Big\}\\ &\quad + \big\{(\,\mathbf{curl}\,\omega, p_h\boldsymbol{\psi}\,) + (\,\mathbf{curl}\,p_h, \omega\boldsymbol{\psi} - (\omega\boldsymbol{\psi})^I\,)\big\} =: G_1 + H_1.\end{aligned} \tag{4.6.17}$$

Setting $\boldsymbol{\psi} = (\omega\boldsymbol{\psi})^I$ in (4.6.1), we get

$$\begin{aligned}
G_1 &= \big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\omega\boldsymbol{\psi})^I \big) - \big(\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}_h - \pi\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi}) \big) \\
&= \big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}(\omega\boldsymbol{\psi}) \big) + \big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}] \big) \\
&\quad - \big(\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}_h - \pi\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi}) \big) \\
&= \Big\{ \big(\mathcal{C}\mathcal{E}(\omega\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi}) \big) - \big(\mathcal{C}\mathcal{E}(\widetilde{\boldsymbol{\phi}_h - \pi\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi}) \big) \Big\} \\
&\quad + \big\{ \mathrm{R}(\omega, \boldsymbol{\psi}, \boldsymbol{\phi}_h) + \big(\mathcal{C}\mathcal{E}(\boldsymbol{\phi}_h), \mathcal{E}[(\omega\boldsymbol{\psi})^I - \omega\boldsymbol{\psi}] \big) \big\} \\
&=: \big(\mathcal{C}\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\boldsymbol{\psi}) \big) + H_2. \qquad (4.6.18)
\end{aligned}$$

Applying the superapproximation property, the Schwarz inequlity, (4.2.7), (4.2.8), and integration by parts, we have

$$\begin{aligned}
&|\ H_1\ | \le C\big(\|p_h\|_{-1,G_h} \|\boldsymbol{\psi}\|_{1,G_h} + h\|p_h\|_{0,G_h} \|\boldsymbol{\psi}\|_{1,G_h} \big), \\
&|\ H_2\ | \le C\big(\|\boldsymbol{\phi}_h\|_{0,G_h} \|\boldsymbol{\psi}\|_{1.G_h} + h\|\boldsymbol{\phi}_h\|_{1,G_h} \|\boldsymbol{\psi}\|_{1,G_h} \big), \\
&|\ (\mathcal{E}(\pi\widetilde{\boldsymbol{\phi}_h}), \mathcal{E}(\boldsymbol{\psi}))\ | \le \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} \|\boldsymbol{\psi}\|_{1,G_h}.
\end{aligned}$$

Combining (4.6.6), (4.6.15)–(4.6.18), the above inequalities, and using the inverse inequalities, we arrive at

$$\begin{aligned}
&\|\pi\widetilde{p_h}\|_{0,G_h} \\
&\le C\big(h\|\boldsymbol{\phi}_h\|_{1,G_h} + \|\boldsymbol{\phi}_h\|_{0,G_h} + ht\|p_h\|_{1,G_h} + h\|p_h\|_{0,G_h} + \|p_h\|_{-1,G_h} \big) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} \\
&\le C\big(\|\boldsymbol{\phi}_h\|_{0,G_h} + ht\|p_h\|_{1,G_h} + \|p_h\|_{-1,G_h} \big) + \|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h}.
\end{aligned} \qquad (4.6.19)$$

Substituting (4.6.19) into (4.6.14) and using the arithmetic-geometric mean inequality, we have

$$\|\pi\widetilde{\boldsymbol{\phi}_h}\|_{1,G_h} + t\|\,\mathbf{curl}\,\pi\widetilde{p_h}\|_{0,G_h} \le C\big(\|\boldsymbol{\phi}_h\|_{0,G_h} + (t^2 + ht)\|p_h\|_{1,G_h} + \|p_h\|_{-1,G_h} \big). \qquad (4.6.20)$$

Substituting (4.6.20) back into (4.6.19), we get

$$\|\pi\widetilde{p_h}\|_{0,G_h} \le C\big(\,\|\boldsymbol{\phi}_h\|_{0,G_h} + \|p_h\|_{-1,G_h} + (ht+t^2)\|p_h\|_{1,G_h}\,\big). \tag{4.6.21}$$

Hence, combining (4.6.20) and (4.6.21) with (4.6.7), we obtain

$$\begin{aligned}&\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\,p_h\|_{0,G_0}\\ &\le C\big(\,\|\boldsymbol{\phi}_h\|_{0,G_h} + \|p_h\|_{-1,G_h} + (ht+t^2)\|p_h\|_{1,G_h}\,\big).\end{aligned} \tag{4.6.22}$$

Applying Theorem 4.5.1 with G_0 replaced by G_1 to bound $\|\boldsymbol{\phi}_h\|_{0,G_1}$ and $\|p_h\|_{-1,G_1}$, we get

$$\begin{aligned}&\|\boldsymbol{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} + t\|\,\mathbf{curl}\,p_h\|_{0,G_0}\\ &\le C\big(\,h\|\boldsymbol{\phi}_h\|_{1,G} + h\|p_h\|_{0,G} + t(t+h)\|p_h\|_{1,G} + \|\boldsymbol{\phi}_h\|_{-\alpha,G} + \|p_h\|_{-\alpha-1,G}\,\big).\end{aligned}$$

Iterating the above inequality $\beta - 1$ times as in section 2.5, or section 3.5, but separating the case $h \le t$ with that $h \ge t$, we prove (4.6.3). $\square$

Theorem 4.6.2. *Let $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Suppose that $(\boldsymbol{\phi}, p) \in \boldsymbol{H}^1 \times H^1$ satisfies $\boldsymbol{\phi}|_{\Omega_1} \in \boldsymbol{H}^2(\Omega_1)$ and $p|_{\Omega_1} \in H^2(\Omega_1)$. Suppose that $(\boldsymbol{\phi}_h, p_h) \in \boldsymbol{V}_h \times P_h$ are such that*

$$(C\mathcal{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}(p-p_h), \boldsymbol{\psi}) = (\boldsymbol{F}, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h, \tag{4.6.23}$$

$$-(\boldsymbol{\phi}-\boldsymbol{\phi}_h, \mathbf{curl}\,q) - \lambda^{-1}t^2(\mathbf{curl}(p-p_h), \mathbf{curl}\,q) = 0 \quad \text{for all } q \in \mathring{P}_h. \tag{4.6.24}$$

for some function $\boldsymbol{F}$ in $\boldsymbol{L}^2$. Let α and β be two arbitrary nonnegative integers. Then, there is a positive number h_1 such that for $h \in (0, h_1]$,

$$\begin{aligned}&\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_0} + \|p-p_h\|_{0,\Omega_0} + t\|\,\mathbf{curl}(p-p_h)\|_{0,\Omega_0}\\ &\le C\big(\,\|\boldsymbol{F}\|_{0,\Omega_1} + h(\|\boldsymbol{\phi}\|_{2,\Omega_1} + \|p\|_{1,\Omega_1} + t\|p\|_{2,\Omega_1}) + t^\beta\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,\Omega_1}\\ &\quad + t^{\beta+1}\|p-p_h\|_{1,\Omega_1} + \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{-\alpha,\Omega_1} + \|p-p_h\|_{-\alpha-1,\Omega_1}\,\big),\end{aligned} \tag{4.6.25}$$

for a constant C depending only on Ω_1, Ω_0, α, and β.

Proof. Let $G_0 \Subset G_0' \Subset G' \Subset G_h \Subset G_1 \Subset G$ be concentric disks and find a $\omega \in C_0^\infty(G')$ with $\omega \equiv 1$ on G_0'. Set $\widetilde{\boldsymbol{\phi}} = \omega\boldsymbol{\phi}$, $\widetilde{p} = \omega p$. Then $\widetilde{\boldsymbol{\phi}} \in \mathring{\boldsymbol{H}}^1(G_h)$, $\widetilde{p} \in H^1(G_h)$. By Lemma 4.4.1, $\pi\widetilde{\boldsymbol{\phi}} \in \mathring{\boldsymbol{V}}_h(G_h)$, $\pi\widetilde{p} \in P_h(G_h)$ can be defined uniquely by the following equations,

$$\big(\, C\mathcal{E}(\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\, \mathbf{curl}(\widetilde{p} - \pi\widetilde{p}), \boldsymbol{\psi}\,\big) = (\boldsymbol{F}, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h), \tag{4.6.26}$$

$$-\big(\, \widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}, \mathbf{curl}\, q\,\big) - \lambda^{-1}t^2\big(\, \mathbf{curl}(\widetilde{p} - \pi\widetilde{p}), \mathbf{curl}\, q\,\big) = 0 \quad \text{for all } q \in P_h(G_h), \tag{4.6.27}$$

with $\int_{G_h} \pi\widetilde{p} = \int_{G_h} \widetilde{p}$. Moreover, we have

$$\begin{aligned} &\|\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_h} + \|\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} + t\|\,\mathbf{curl}(\widetilde{p} - \pi\widetilde{p})\|_{0,G_h} \\ &\le C\big(\inf_{q \in P_h(G_h)} (\|\widetilde{p} - q\,\|_{\,0,G_h} + t\|\,\mathbf{curl}(\widetilde{p} - q)\|_{0,G_h}) + \|\boldsymbol{F}\|_{0,G_h}\,\big) \\ &\qquad + \inf_{\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_h)} \|\widetilde{\boldsymbol{\phi}} - \boldsymbol{\psi}\|_{1,G_h}\,\big) \\ &\le C\big(\,\|\boldsymbol{\phi}\|_{1,G} + \|p\|_{0,G} + t\|p\|_{1,G} + \|\boldsymbol{F}\|_{0,G}\,\big). \end{aligned} \tag{4.6.28}$$

Let us now estimate $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0}$, $\|p - p_h\|_{0,G_0}$, and $t\|\,\mathbf{curl}(p - p_h)\|_{0,G_0}$. By the triangle inequality, we have

$$\begin{aligned} &\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0} + t\|\,\mathbf{curl}(p - p_h)\|_{0,G_0} \\ &\le \|\boldsymbol{\phi} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0} + \|p - \pi\widetilde{p}\|_{0,G_0} + \|\pi\widetilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_h\|_{1,G_0} + \|\pi\widetilde{p} - p_h\|_{0,G_0} \\ &\quad + t\|\,\mathbf{curl}(p - \pi\widetilde{p})\|_{0,G_0} + t\|\,\mathbf{curl}(\pi\widetilde{p} - p_h)\|_{0,G_0}. \end{aligned} \tag{4.6.29}$$

Since (4.6.26), (4.6.27) and (4.6.23), (4.6.24) hold for any $\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_0')$,

$q \in \mathring{P}_h(G_0')$, respectively. Subtracting the corresponding two equations, we obtain

$$(\mathcal{E}(\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}), \mathcal{E}(\boldsymbol{\psi})) - (\mathbf{curl}(p_h - \pi\widetilde{p}), \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h(G_0'), \tag{4.6.30}$$

$$-(\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}, \mathbf{curl}\, q) - \lambda^{-1}t^2(\mathbf{curl}(p_h - \pi\widetilde{p}), \mathbf{curl}\, q) = 0 \quad \text{for all } q \in \mathring{P}_h(G_0'). \tag{4.6.31}$$

Then we apply Lemma 4.6.1 to $\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}$ and $p_h - \pi\widetilde{p}$ with G replaced by G_0' to obtain

$$\begin{aligned}
&\|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0} + \|p_h - \pi\widetilde{p}\|_{0,G_0} + t\|\,\mathbf{curl}(p_h - \pi\widetilde{p})\|_{0,G_0} \\
&\leq C\big(\, t^\beta\|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_0'} + t^{\beta+1}\|p_h - \pi\widetilde{p}\|_{1,G_0'} + \|\boldsymbol{\phi}_h - \pi\widetilde{\boldsymbol{\phi}}\|_{-\alpha,G_0'} \\
&\qquad + \|p_h - \pi\widetilde{p}\|_{-\alpha-1,G_0'}\,\big) \\
&\leq C\big(\, t^{\beta+1}\|p - \pi\widetilde{p}\|_{1,G_0'} + t^{\beta+1}\|p_h - p\|_{1,G_0'} + \|\boldsymbol{\phi} - \pi\widetilde{\boldsymbol{\phi}}\|_{-\alpha,G_0'} + \|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_{-\alpha,G_0'} \\
&\quad + t^\beta\|\boldsymbol{\phi} - \pi\boldsymbol{\phi}_h\|_{1,G_0'} + t^\beta\|\boldsymbol{\phi}_h - \boldsymbol{\phi}_h\|_{1,G_0'} + \|p - \pi\widetilde{p}\|_{-\alpha-1,G_0'} + \|p - p_h\|_{-\alpha-1,G_0'}\,\big) \\
&\leq C\big(\, \|\widetilde{\boldsymbol{\phi}} - \pi\widetilde{\boldsymbol{\phi}}\|_{1,G_h} + \|\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} + t\|\,\mathbf{curl}(\widetilde{p} - \pi\widetilde{p}\|_{0,G_h} \\
&\quad + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G} + t^{\beta+1}\|p - p_h\|_{1,G} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G} + \|p - p_h\|_{-\alpha-1,G}\,\big).
\end{aligned} \tag{4.6.32}$$

Combining (4.6.28), (4.6.29), and (4.6.32), we obtain

$$\begin{aligned}
&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0} + t\|\,\mathbf{curl}(p - p_h)\|_{0,G_0} \\
&\leq C\big(\, \|\boldsymbol{F}\|_{0,G} + \|\boldsymbol{\phi}\|_{1,G} + \|p\|_{0,G} + t\|\,\mathbf{curl}\, p\|_{0,G} \\
&\qquad + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G} + t^{\beta+1}\|p - p_h\|_{1,G} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G} + \|p - p_h\|_{-\alpha-1,G}\,\big).
\end{aligned}$$

Since $[(\boldsymbol{\phi} - q) - (\boldsymbol{\phi}_h - q)]$ and $[(p - q) - (p_h - q)]$ also satisfy equations (4.6.23) and (4.6.24) for any $\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h$ and $q \in \mathring{P}_h$, we have

$$\begin{aligned}
&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0} + t\|\,\mathbf{curl}(p - p_h)\|_{0,G_0} \\
&\leq C\big(\inf_{\boldsymbol{\psi}\in\mathring{\boldsymbol{V}}_h} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,G} + \inf_{q\in\mathring{P}_h} (\|p - q\|_{0,G} + t\|\,\mathbf{curl}(p - q)\|_{0,G}) + \|\boldsymbol{F}\|_{0,G} \\
&\qquad + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,G} + \|p - p_h\|_{-\alpha-1,G} + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G} + t^{\beta+1}\|p - p_h\|_{1,G}\,\big).
\end{aligned}$$

Then, first using the approximation properties of the finite element spaces and then a covering argument (cf. section 2.5 and section 3.5), we obtain the desired result. □

We now state the main result of this chapter.

Theorem 4.6.3. *Let* $\Omega_0 \Subset \Omega_1 \Subset \Omega$ *and suppose that* $(r, \boldsymbol{\phi}, p, w) \in H^1 \times \boldsymbol{H}^1 \times H^1 \times H^1$ *(the exact solution) satisfies* $(r, \boldsymbol{\phi}, p, w)|_{\Omega_1} \in H^2(\Omega_1) \times \boldsymbol{H}^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1)$*. Suppose that* $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in W_h \times \boldsymbol{V}_h \times P_h \times W_h$ *(the finite element solution) is given so that* (4.5.1), (4.5.2), (4.5.3), *and* (4.5.4) *hold. Let* α, β *be two nonnegative integers. Then there exists a positive number* h_1 *and a constant* C *depending only on* Ω_0, Ω_1, α, *and* β, *such that for all* $h \in (0, h_1]$

$$\|r - r_h\|^h_{1,\Omega_0} \le C\big(\, h\|r\|_{2,\Omega_1} + \|r - r_h\|_{-\alpha,\Omega_1}\,\big) \tag{4.6.33}$$

$$\begin{aligned}
&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_0} + \|p - p_h\|_{0,\Omega_0} + t\|\,\mathbf{curl}(p - p_h)\|_{0,\Omega_0} \\
&\le C\big(\, h(\|\boldsymbol{\phi}\|_{2,\Omega_1} + \|p\|_{1,\Omega_1} + t\|p\|_{2,\Omega_1} + \|r\|_{2,\Omega_1}) + \|r - r_h\|_{-\alpha,\Omega_1} \\
&\quad + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,\Omega_1} + \|p - p_h\|_{-\alpha-1,\Omega_1} + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_1} + t^{\beta+1}\|p - p_h\|_{1,\Omega_1}\,\big),
\end{aligned} \tag{4.6.34}$$

$$\begin{aligned}
\|w - w_h\|^h_{1,\Omega_0} \le \big(\, &Ch(\|\boldsymbol{\phi}\|_{2,\Omega_1} + \|p\|_{1,\Omega_1} + t\|p\|_{2,\Omega_1} + \|r\|_{2,\Omega_1} + \|w\|_{2,\Omega_1}) \\
&+ \|w - w_h\|_{-\alpha,\Omega_1} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-\alpha,\Omega_1} + \|r - r_h\|_{-\alpha,\Omega_1} \\
&+ \|p - p_h\|_{-\alpha-1,\Omega_1} + t^\beta\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_1} + t^{\beta+1}\|p - p_h\|_{1,\Omega_1}\,\big).
\end{aligned} \tag{4.6.35}$$

Proof. Find a subdomain Ω' such that $\Omega_0 \Subset \Omega' \Subset \Omega_1$. Apllying Theorem 2.5.2 with Ω_0 replaced by Ω' yields

$$\|r - r_h\|^h_{1,\Omega'} \le C\big(\, h\|r\|_{2,\Omega_1} + \|r - r_h\|_{-\alpha,\Omega_1}\,\big), \tag{4.6.36}$$

which also implies (4.6.33). From (4.5.2), (4.5.3), (4.6.36), and Theorem 4.6.2 with

Ω_1 replaced by Ω', we obtain

$$\begin{aligned}
&\|\phi-\phi_h\|_{1,\Omega_0}+\|p-p_h\|_{0,\Omega_0}+t\|\,\mathbf{curl}(p-p_h)\|_{0,\Omega_0}\\
&\le C\big(\,h(\|\phi\|_{2,\Omega'}+\|p\|_{1,\Omega'}+t\|p\|_{2,\Omega'})+\|\,\mathbf{grad}_h(r-r_h)\|_{0,\Omega'}\\
&\quad+\|\phi-\phi_h\|_{-\alpha,\Omega'}+\|p-p_h\|_{-\alpha-1,\Omega'}+t^{\beta+1}\|p-p_h\|_{1,\Omega'}+t^{\beta}\|\phi-\phi_h\|_{1,\Omega'}\,\big)\\
&\le C\big(\,h(\|\phi\|_{2,\Omega_1}+\|p\|_{1,\Omega_1}+t\|p\|_{2,\Omega_1}+\|r\|_{2,\Omega_1})+\|\phi-\phi_h\|_{-\alpha,\Omega_1}\\
&\quad+\|p-p_h\|_{-\alpha-1,\Omega_1}+t^{\beta}\|\phi-\phi_h\|_{1,\Omega_1}+t^{\beta+1}\|p-p_h\|_{1,\Omega_1}+\|r-r_h\|_{-\alpha,\Omega_1}\,\big).
\end{aligned}$$

This completes (4.6.34).

We now consider the interior estimate for the transverse displacement. Because of the difference between (2.4.1) (required by Theorem 2.5.2)) and (4.5.4) (satisfied by $(w-w_h)$), Theorem 2.5.2 cannot be used directly. But we can follow the same proof to get (4.6.35).

Let $G_0 \Subset G_1 \Subset G$ be concentric disks and G_h be a union of triangles which satisfies $G_1 \Subset G_h \Subset G$. Use the notation $\widetilde{w}=\omega w$ and define $\pi\widetilde{w}\in \bar{W}_h(G_h)$ by

$$(\mathbf{grad}_h\,\pi\widetilde{w},\mathbf{grad}_h\,s)=(\mathbf{grad}_h\,\widetilde{w},\mathbf{grad}_h\,s)\quad\text{for all } s\in\bar{W}_h(G_h).$$

We have

$$\|\widetilde{w}-\pi\widetilde{w}\|^h_{1,G_h}\le C\inf_{s\in\bar{W}_h(G_h)}\|\widetilde{w}-s\|_{1,G_h}\le C\,h\|w\|_{2,G_h}.$$

By the triangle inequality,

$$\begin{aligned}
\|w-w_h\|^h_{1,G_0}&\le\|\widetilde{w}-\pi\widetilde{w}\|^h_{1,G_h}+\|\pi\widetilde{w}-w_h\|^h_{1,G_0}\\
&\le Ch\|w\|_{2,G_h}+\|\pi\widetilde{w}-w_h\|^h_{1,G_0}.
\end{aligned}\tag{4.6.37}$$

From (4.5.4) and the fact that $\omega=1$ on G_1,

$$\begin{aligned}
(\mathbf{grad}_h(\pi\widetilde{w}-w_h),\mathbf{grad}_h\,s)&=(\mathbf{grad}_h(w-w_h),\mathbf{grad}_h\,s)\\
&=L(s)\quad\text{for all } s\in\mathring{W}_h(G_1),
\end{aligned}\tag{4.6.38}$$

where

$$L(s) := (\boldsymbol{\phi} - \boldsymbol{\phi}_h + \lambda^{-1}t^2 \,\mathbf{grad}_h(r - r_h), \mathbf{grad}_h\, s) - \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\phi} \cdot \boldsymbol{n}_T + \lambda^{-1}t^2 \frac{\partial r}{\partial n}) s,$$

for all $s \in \mathring{W}_h$. By Lemma 2.3.2,

$$\begin{aligned} |L(s)_{G_h}| \le C\big(&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_h} + \|r - r_h\|^h_{1,G_h} \\ &+ h(\|\boldsymbol{\phi}\|_{1,G_h} + \|r\|_{2,G_h})\big)\|\,\mathbf{grad}_h\, s\|_{0,G_h} \quad \text{for all } s \in \mathring{W}_h(G_h), \end{aligned}$$

which implies

$$\|L\|_{G_h} \le C\big(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G_h} + \|r - r_h\|^h_{1,G_h} + h(\|\boldsymbol{\phi}\|_{1,G_h} + \|r\|_{2,G_h})\big).$$

Under (4.6.38) we apply Lemma 2.5.1 with G replaced by G_1 to obtain

$$\begin{aligned} \|\pi\widetilde{w} - w_h\|^h_{1,G_0} &\le C\big(\|\pi\widetilde{w} - w_h\|_{-t,G_1} + \|L\|_{G_1}\big) \\ &\le C(\|\pi\widetilde{w} - \widetilde{w}\|_{-t,G_h} + \|w - w_h\|_{-t,G_1} + \|L\|_{G_h}) \\ &\le C\big(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G'} + \|r - r_h\|^h_{1,G'} + \|w - w_h\|_{-\alpha,G} \\ &\quad + h(\|\boldsymbol{\phi}\|_{1,G} + \|r\|_{2,G})\big). \end{aligned} \tag{4.6.39}$$

Using the triangle inequality, (4.6.37), and (4.6.39) yields

$$\begin{aligned} \|w - w_h\|^h_{1,G_0} \le C\big(&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,G'} + \|r - r_h\|^h_{1,G'} + \|w - w_h\|_{-\alpha,G} \\ &+ h(\|\boldsymbol{\phi}\|_{1,G} + \|w\|_{2,G} + \|r\|_{2,G})\big). \end{aligned}$$

Applying (4.6.33) and (4.6.34) with Ω_0 and Ω_1 replaced by G' and G, respectively, we obtain a local version of (4.6.35). Then a standard covering argument leads to (4.6.35). $\square$

4.7 The Global and Interior Convergences of the Arnold-Falk Element

As an application of the theory we developed in the last section, we consider the soft simply supported plate with a smooth forcing function g. Under this boundary condition (for a smooth Ω), the exact solution of the Reissner-Mindlin plate satisfies (cf. [5])

$$\begin{gathered}\|r\|_2 + \|w\|_2 + \|\phi\|_{3/2} + \|p\|_{1/2} + t\|p\|_{3/2} \le C, \\ \|\phi\|_{3/2+\epsilon} + \|p\|_{1/2+\epsilon} + t\|p\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon}, \\ \|\phi\|_2 + \|p\|_1 + t\|p\|_2 \le Ct^{-1/2},\end{gathered} \tag{4.7.1}$$

for a constant C_ϵ that is independent of t and h, and $\epsilon \in (0, 1/2]$. Obviously, functions ϕ and p are not regular enough to ensure that the MINI element converges at the optimal rate, uniformly in the plate thickness t.

Thus, we want to use Theorem 4.6.3 to obtain the interior convergence rates of the Arnold-Falk element. To do so, we must estimate $\|p - p_h\|_1$, $\|\phi - \phi_h\|_1$, $\|r - r_h\|_{-\alpha,\Omega_1}$, $\|\phi - \phi_h\|_{-\alpha,\Omega_1}$, $\|p - p_h\|_{-\alpha-1,\Omega_1}$, and $\|w - w_h\|_{-\alpha,\Omega_1}$ for some suitable domain Ω_1 and integer α. The only way of doing these, as far as we know, is to use the inequality $\|\cdot\|_{t,\Omega_1} \le \|\cdot\|_{t,\Omega}$. Hence, the global convergence of the nonconforming element for the Poisson equation and that of the MINI element for the Stokes-like equations must be established. The first one is well-known but the second one is difficult due to the special structure of the singularly perturbed Stokes-like equations and the type of the boundary condition imposed. Because of the difficulty in dealing with the boundary approximation when the boundary layer exists, we will only work on a polygonal domain, i.e., we will assume that Ω is a convex polygon. Therefore, we need to know the regularity of the exact solution of the singularly perturbed generalized Stokes-like equations on a convex polygon (under the soft simply supported boundary condition). But so far we cannot prove the regularity result (Theorem 4.7.2) we need. So we will assume that it is true. We feel that we have reason to believe it is correct (possibly with some restriction on the magnitude of the maximum angle of the polygon). As a partial justification, we will prove a similar result for a smooth Ω in Appendix B.

This section is organized as follows. Theorem 4.7.1 presents a technical result on the approximation property of the continuous piecewise linear functions. Its proof can be found in Appendix A. Theorem 4.7.2 is the assumption we just mentioned. The global convergence of the Arnold-Falk element is given in Theorem 4.7.3. Finally by combing Theorem 4.7.2 and Theorem 4.6.2 we get the interior convergence rate of the Arnold-Falk element for the rotation (Theorem 4.7.4).

Theorem 4.7.1. *Let Ω be a convex polygon and $u \in H^2$. Then there exists an operator $\pi_h : H^1 \to P_h$ such that*

$$\|p - \pi_h p\|_0 \leq C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}, \tag{4.7.2}$$

$$\|p - \pi_h p\|_1 \leq C h^{1/2} \|p\|_{3/2}, \tag{4.7.3}$$

$$\|p - \pi_h p\|_{-1/2,\partial\Omega} \leq C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}, \tag{4.7.4}$$

for any $0 < \epsilon \leq 1/2$. Here C is independent of ϵ and C_ϵ depends on ϵ, but not h.

Proof. See Theorem 5.3.1, Appendix A.

Theorem 4.7.2. *Let Ω be a convex polygon and $\boldsymbol{F} \in \boldsymbol{H}^1$ and $K \in H^2 \cap \hat{L}^2$. Then there exists a unique solution $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^2 \times H^1 \cap \hat{L}^2$ to the equations*

$$(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\boldsymbol{\Psi}, \mathbf{curl}\, P) = (\boldsymbol{F}, \boldsymbol{\Psi}) \quad \text{for all } \boldsymbol{\Psi} \in \boldsymbol{H}^1, \tag{4.7.5}$$

$$-(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - \lambda^{-1} t^2 (\mathbf{curl}\, Q, \mathbf{curl}\, P) = (K, Q) \quad \text{for all } Q \in H^1. \tag{4.7.6}$$

Moreover,

$$\|\mathbf{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon}\big(\,\|\boldsymbol{F}\|_0 + t^{3/2}\|\boldsymbol{F}\|_1\,\big) \quad \textit{for} \quad K = 0, \tag{4.7.7}$$

$$\|\mathbf{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon}\|K\|_2 \quad \textit{for} \quad \boldsymbol{F} = 0, \tag{4.7.8}$$

for $0 \le \epsilon \le 1/2$.

Proof. See Corollary 6.4, Appendix B.

Theorem 4.7.3. *Let* Ω *be a convex polygon. Assume that* $(r, \boldsymbol{\phi}, p, w)$ *and* $(r_h, \boldsymbol{\phi}_h, p_h, w_h)$ *solve* (4.2.3)-(4.2.6) *and* (4.4.4)-(4.4.7)*, respectively, for some smooth* g*, some* $t \in (0, 1]$*, and a quasi-uniform mesh* $\mathcal{T}_h$*. Then,*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|p - p_h\|_0 + t\|\,\mathbf{curl}(p - p_h)\|_0 \le C_\epsilon h^{1/2} t^{-\epsilon}, \tag{4.7.9}$$

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|w - w_h\|_1^h \le C_\epsilon h t^{-\epsilon}, \tag{4.7.10}$$

$$t\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t^2\|\,\mathbf{curl}(p - p_h)\|_0 \le Ch, \tag{4.7.11}$$

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-1} + \|p - p_h\|_{-2} \le C_\epsilon h t^{-\epsilon}, \tag{4.7.12}$$

for an arbitrarily small constant ϵ.

Proof. Subtracting (4.2.4) by (4.4.5) and (4.2.5) by (4.4.6), respectively, we obtain

$$\big(\,\mathcal{C}\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,\boldsymbol{\psi}, \mathbf{curl}(p - p_h)\,\big) = \big(\,\mathbf{grad}(r - r_h), \boldsymbol{\psi}\,\big) \text{ for all } \boldsymbol{\psi} \in \boldsymbol{V}_h \tag{4.7.13}$$

$$-\big(\,\mathbf{curl}\, q, \boldsymbol{\phi} - \boldsymbol{\phi}_h\,\big) - \lambda^{-1}t^2\big(\,\mathbf{curl}(p - p_h), \mathbf{curl}\, q\,\big) = 0 \text{ for all } q \in P_h \tag{4.7.14}$$

From (4.7.13) and (4.7.14) (see also the proof on page 1284 of [3, Theorem 5.5]), we have

$$\begin{aligned}
&\|\mathcal{C}^{\frac{1}{2}}\mathcal{E}(\boldsymbol{\phi}_h - \boldsymbol{\psi})\|_0^2 + \lambda^{-1}t^2\|\,\mathbf{curl}(p_h - q)\|_0^2 \\
= &\big(\,\mathcal{C}\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\psi}), \mathcal{E}(\boldsymbol{\phi}_h - \boldsymbol{\psi})\,\big) + \lambda^{-1}t^2\big(\,\mathbf{curl}(p - q), \mathbf{curl}(p_h - q)\,\big) \\
&- \big(\,\mathbf{curl}(p - q), \boldsymbol{\phi}_h - \boldsymbol{\psi}\,\big) + \big(\,\boldsymbol{\phi} - \boldsymbol{\psi}, \mathbf{curl}(p_h - q)\,\big) + \big(\,\mathbf{grad}_h(r_h - r), \boldsymbol{\phi}_h - \boldsymbol{\psi}\,\big).
\end{aligned} \tag{4.7.15}$$

In the above we choose $\boldsymbol{\psi}$ to be the Fortin projection of $\boldsymbol{\phi}$, that is,

$$(\boldsymbol{\phi} - \boldsymbol{\Pi\phi}, \mathbf{curl}\, q) = 0 \qquad \text{for all } q \in P_h,$$
$$\|\boldsymbol{\phi} - \boldsymbol{\Pi\phi}\|_1 \le C\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_1 \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{V}_h.$$

We see that the fourth term on the right hand side of (4.7.15) is gone. Taking $q = \pi_h p$ to be the interpolant of p described in Theorem 4.7.1 and using the Schwartz inequality, integration by parts, and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
&\|C^{\frac{1}{2}}\mathcal{E}(\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi})\|_0^2 + \lambda^{-1}t^2\|\,\mathbf{curl}(p_h - \pi_h p)\|_0^2 \\
&\le C(\|\boldsymbol{\phi} - \boldsymbol{\Pi\phi}\|_1^2 + t^2\|p - \pi_h p\|_1^2 + \|p - \pi_h p\|_0\|\operatorname{rot}(\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi})\|_0 \\
&\quad + \|\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi}\|_1\|p - \pi_h p\|_{-1/2,\partial\Omega} + h\|\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi}\|_1).
\end{aligned} \tag{4.7.16}$$

To estimate $\|\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi}\|_1$, we note that the $\boldsymbol{H}^1$ norm of the vector function $\boldsymbol{\phi}_h - \boldsymbol{\Pi\phi}$ is equivalent to

$$\|\mathcal{E}(\boldsymbol{\Pi\phi} - \boldsymbol{\phi}_h)\|_0 + |\int \boldsymbol{\Pi\phi} - \boldsymbol{\phi}_h| + |\int(\boldsymbol{\Pi\phi} - \boldsymbol{\phi}_h)\cdot(-y, x)|. \tag{4.7.17}$$

The first term in (4.7.17) is already covered by (4.7.16). To control the other two, we take $q = y$ and $q = x$, respectively in (4.7.14) to get

$$|\int(\boldsymbol{\phi} - \boldsymbol{\phi}_h)| \le Ct^2|\int \mathbf{curl}(p - p_h)|,$$

which implies

$$\begin{aligned}
&|\int(\boldsymbol{\Pi\phi} - \boldsymbol{\phi}_h)| \le |\int(\boldsymbol{\Pi\phi} - \boldsymbol{\phi})| + Ct^2\|\,\mathbf{curl}(p - p_h)\|_0 \\
&\le C\big(\,\|\boldsymbol{\phi} - \boldsymbol{\Pi\phi}\|_0 + t^2\|\,\mathbf{curl}(p - \pi_h p)\|_0 + t^2\|\,\mathbf{curl}(\pi_h p - p_h)\|_0\,\big).
\end{aligned} \tag{4.7.18}$$

To control the third term in (4.7.17), we take $q = \mathcal{L}_h q_0 = \mathcal{L}_h(x^2+y^2)/2$: the L^2 projection of $(x^2+y^2)/2$ in $P_h(\Omega)$ in (4.7.14), to obtain

$$\big(\,\boldsymbol{\phi}-\boldsymbol{\phi}_h, \mathbf{curl}\, q_0\,\big) = \big(\,\boldsymbol{\phi}-\boldsymbol{\phi}_h, \mathbf{curl}(q_0-\mathcal{L}_h q_0)\,\big) \\ -\lambda^{-1}t^2\big(\,\mathbf{curl}(p-p_h), \mathbf{curl}\,\mathcal{L}_h q_0\,\big).$$

Using the fact that

$$\|q_0-\mathcal{L}_h q_0\|_{1,\infty} \le Ch$$

and

$$\|\mathcal{L}_h q_0\|_{1,\infty} \le C,$$

we obtain

$$\Big|\int(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\cdot(-y,x)\Big| \le C\big(\,h\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0 + t^2\Big|\int \mathbf{curl}(p-p_h)\Big|\,\big). \tag{4.7.19}$$

Therefore,

$$\Big|\int(\mathbf{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi}_h)\cdot(-y,x)\Big| \\ \le C\big(\,\|\boldsymbol{\phi}-\mathbf{\Pi}\boldsymbol{\phi}\|_0 + h\|\mathbf{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0 + t^2\|\,\mathbf{curl}(p-\pi_h p)\|_0 + t^2\|\,\mathbf{curl}(\pi_h p-p_h)\|_0\,\big). \tag{4.7.20}$$

Combining (4.7.16)–(4.7.20) and using the arithmetic-geometric mean inequality, we get

$$\|\boldsymbol{\phi}_h-\mathbf{\Pi}\boldsymbol{\phi}\|_1 + t\|\,\mathbf{curl}(p_h-\pi_h p)\|_0 \\ \le C\big(\,\|\boldsymbol{\phi}-\mathbf{\Pi}\boldsymbol{\phi}\|_1 + t\|p-\pi_h p\|_1 + \|p-\pi_h p\|_0 + \|p-\pi_h p\|_{-1/2,\partial\Omega} + h\,\big).$$

Applying the approximation property of $\mathbf{\Pi}$ and Theorem 4.7.1, we get

$$\|\boldsymbol{\phi}_h-\mathbf{\Pi}\boldsymbol{\phi}\|_1 + t\|\,\mathbf{curl}(p_h-\pi_h p)\|_0 \\ \le C_\epsilon h^{1/2}\big(\,\|\boldsymbol{\phi}\|_{3/2} + t\|p\|_{3/2} + \|p\|_{1/2+\epsilon} + h^{1/2}\,\big),$$

which implies

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\,\mathbf{curl}(p - p_h)\|_0 \le C_\epsilon h^{1/2} t^{-\epsilon}. \qquad (4.7.21)$$

Moreover,

$$\begin{aligned} t\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t^2\|\,\mathbf{curl}(p - p_h)\|_0 &\le Ct\big(\,\|\boldsymbol{\phi} - \boldsymbol{\Pi}\boldsymbol{\phi}\|_1 + t\|p - \pi_h p\|_1 \\ &\qquad + \|p - \pi_h p\|_{-1/2,\partial\Omega} + h\,\big) \\ &\le Cth\big(\,\|\boldsymbol{\phi}\|_2 + t\|p\|_2 + 1\,\big) \le Ch. \end{aligned}$$

To estimate $\|p - p_h\|_0$, we simply repeat the proof on page 1285 of [3, Theorem 5.5]. By the stability condition, there exits $\gamma > 0$ independent of h such that for all $q \in \hat{W}_h$ there exists a nonzero $\boldsymbol{\psi} \in \mathring{\boldsymbol{V}}_h$ with

$$\gamma\|q\|_0\|\,\mathbf{grad}\,\boldsymbol{\psi}\|_0 \le (\mathbf{curl}\,q, \boldsymbol{\psi}).$$

Applying this result with q replaced by $(p_h - \pi_h p + \int_\Omega \pi_h p)$, and again using (4.2.4) and (4.4.6), we have

$$\begin{aligned} \gamma\|p_h - \pi_h p + \int_\Omega \pi_h p\|_0\|\,\mathbf{grad}\,\boldsymbol{\psi}\|_0 &\le \big(\,\mathbf{curl}(p_h - \pi_h p), \boldsymbol{\psi}\,\big) \\ = \big(\,\mathbf{curl}(p - \pi_h p), \boldsymbol{\psi}\,\big) &+ \big(\,C\mathcal{E}(\boldsymbol{\phi}_h - \boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi})\,\big) - \big(\,\mathbf{grad}_h(r_h - r), \boldsymbol{\psi}\,\big) \\ \le C\big(\,\|p - \pi_h p\|_0 &+ \|\,\mathbf{grad}(\boldsymbol{\phi}_h - \boldsymbol{\phi})\|_0 + \|\,\mathbf{grad}_h(r_h - r)\|_0\,\big)\|\,\mathbf{grad}\,\boldsymbol{\psi}\|_0, \end{aligned}$$

so

$$\|p_h - \pi_h p + \int_\Omega \pi_h p\|_0 \le C\big(\,\|p - \pi_h p\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\,\mathbf{grad}_h(r - r_h)\|_0\,\big).$$

By the triangle inequality, we get

$$\|p - p_h\|_0 \le C\big(\,\|p\|_0 + \|p - \pi_h p\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\,\mathbf{grad}_h(r - r_h)\|_0\,\big).$$

Since if $(\boldsymbol{\phi} - \boldsymbol{\phi}_h, p - p_h)$ satisfies (4.7.13) and (4.7.14), so will $[(\boldsymbol{\phi} - \boldsymbol{\psi}) - (\boldsymbol{\phi}_h - \boldsymbol{\psi}), (p - q) - (p_h - q)]$, for any $(\boldsymbol{\psi}, q) \in \boldsymbol{V}_h \times W_h$. Therefore, together with (4.7.21) and Theorem 4.7.1, we get

$$\|p - p_h\|_0 \leq C\big(\|p - \pi_h p\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\,\mathbf{grad}_h(r - r_h)\|_0 \,\big) \leq C_\epsilon h^{1/2} t^{-\epsilon}.$$

This completes (4.7.9). To bound $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0$, we construct the following duality problem:

Find $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^1 \times \hat{H}^1$ such that

$$(\,C\mathcal{E}(\boldsymbol{\psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\boldsymbol{\psi}, \mathbf{curl}\, P) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}^1, \tag{4.7.22}$$

$$-(\mathbf{curl}\, q, \boldsymbol{\Phi}) - \lambda^{-1} t^2 (\mathbf{curl}\, q, \mathbf{curl}\, P) = 0 \quad \text{for all } q \in H^1. \tag{4.7.23}$$

From Corollary 4.2.4 we know that $(\boldsymbol{\Phi}, P)$ is uniquely defined. Moreover,

$$\|\boldsymbol{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \leq C_\epsilon t^{-\epsilon} \big(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + t^{3/2} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 \,\big) \tag{4.7.24}$$

Following the proof of Theorem 6.1 on page 1286 of [3], we can obtain, for $(q, \boldsymbol{\psi}) = (\pi_h P, \boldsymbol{\Pi\Phi})$, where $\boldsymbol{\Pi}$ is the Fortin projection and π_h the interpolant as described in Theorem 4.7.1,

$$\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0^2 =& \big(\, C\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\Phi} - \boldsymbol{\Pi\Phi}) \,\big) - \big(\, \boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{curl}(P - \pi_h P) \,\big) \\
& - \big(\, \mathbf{curl}(p - \pi_h p), \boldsymbol{\Phi} - \boldsymbol{\Pi\Phi} \,\big) - \lambda^{-1} t^2 \big(\, \mathbf{curl}(p - p_h), \mathbf{curl}(P - \pi_h P) \,\big) \\
& + \big(\, \mathbf{grad}_h(r - r_h), \boldsymbol{\Pi\Phi} \,\big) \\
=& \big(\, C\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\boldsymbol{\Phi} - \boldsymbol{\Pi\Phi}) \,\big) - \big(\, \mathrm{rot}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), P - \pi_h P \,\big) \\
& - \langle (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \cdot \boldsymbol{s}, P - \pi_h P \rangle_{\partial\Omega} + \big(\, p - \pi_h p, \mathrm{rot}(\boldsymbol{\Phi} - \boldsymbol{\Pi\Phi}) \,\big) \\
& - \langle p - \pi_h p, (\boldsymbol{\Phi} - \boldsymbol{\Pi\Phi}) \cdot \boldsymbol{s} \rangle_{\partial\Omega} - \lambda^{-1} t^2 \big(\, \mathbf{curl}(p - p_h), \mathbf{curl}(P - \pi_h P) \,\big) \\
& + \big(\, \mathbf{grad}_h(r - r_h), \boldsymbol{\Pi\Phi} \,\big)
\end{aligned} \tag{4.7.25}$$

Using the Schwartz inequality and integration by parts in (4.7.25), we get

$$\begin{aligned}
&\|\phi-\phi_h\|_0^2 \\
&\le C\|\phi-\phi_h\|_1(\|\mathbf{\Phi}-\mathbf{\Pi\Phi}\|_1+\|P-\pi_hP\|_0+\|P-\pi_hP\|_{-1/2,\partial\Omega}) \\
&\quad+\|\mathbf{\Phi}-\mathbf{\Pi\Phi}\|_1(\|p-\pi_hp\|_0+\|p-\pi_hp\|_{-1/2,\partial\Omega}) \\
&\quad+t^2\|\,\mathbf{curl}(p-p_h)\|_0\|\,\mathbf{curl}(P-\pi_hP)\|_0+\|\,\mathbf{grad}_h(r-r_h)\|_0\|\mathbf{\Pi\Phi}\|_0.
\end{aligned} \tag{4.7.26}$$

If $t\le h$, then by Theorem 4.7.1, (4.7.26), (4.7.24), and (4.7.9), we obtain

$$\begin{aligned}
&\|\phi-\phi_h\|_0^2 \\
&\le C_\epsilon ht^{-\epsilon}\big(\,\|\mathbf{\Phi}\|_{3/2}+\|P\|_{1/2+\epsilon}\big)+Ch\|\mathbf{\Phi}\|_{3/2}\|p\|_{1/2+\epsilon}+C_\epsilon h\|P\|_{3/2}+Ch\|\mathbf{\Phi}\|_1 \\
&\le C_\epsilon ht^{-2\epsilon}\big(\,\|\phi-\phi_h\|_0+t^{3/2}\|\phi-\phi_h\|_1\,\big) \\
&\le C_\epsilon ht^{-2\epsilon}\big(\,\|\phi-\phi_h\|_0+h^2\,\big),
\end{aligned}$$

which implies

$$\|\phi-\phi_h\|_0\le C_\epsilon ht^{-2\epsilon}.$$

If $t\ge h$, using (4.7.24) with $\epsilon=1/2$ in (4.7.26), we obtain

$$\begin{aligned}
&\|\phi-\phi_h\|_0^2 \\
&\le C_\epsilon h^{3/2}t^{-\epsilon}(\|\mathbf{\Phi}\|_2+\|P\|_1)+C_\epsilon h^{3/2}t^{-\epsilon}\|\mathbf{\Phi}\|_2+C_\epsilon h^{3/2}t^{-\epsilon}\|P\|_2+Ch\|\mathbf{\Phi}\|_1 \\
&\le C_\epsilon h^{3/2}t^{-\epsilon}t^{-1/2}(\|\phi-\phi_h\|_0+t^{3/2}\|\phi-\phi_h\|_1)+Ch\|\phi-\phi_h\|_0.
\end{aligned}$$

Since $t\ge h$,

$$\|\phi-\phi_h\|_0^2\le C_\epsilon ht^{-\epsilon}(\|\phi-\phi_h\|_0+h^2).$$

Hence

$$\|\phi-\phi_h\|_0\le C_\epsilon ht^{-\epsilon}.$$

To analyze $w - w_h$, we follow that in [3,Theorem 5.5]. Define $\bar{\omega}_h \in \bar{W}_h$ by

$$\left(\mathbf{grad}_h\, \bar{w}_h, \mathbf{grad}_h\, s \right) = \left(\boldsymbol{\phi} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}_h\, s \right) \quad \text{for all } s \in \bar{W}_h.$$

So

$$\|w - \bar{w}_h\|_1^h \le Ch\big(\|\boldsymbol{\phi} + t^2 r\|_1 + \|w\|_2 \big) \le Ch,$$

where we use the fact that $\|\boldsymbol{\phi}\|_1 + \|r\|_1 + \|w\|_2 \le C$, with C independent of t. In addition, we have

$$\left(\mathbf{grad}_h(\bar{w}_h - w_h), \mathbf{grad}_h\, s \right) = \left(\boldsymbol{\phi} - \boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}_h(r - r_h), \mathbf{grad}_h\, s \right),$$

for all $s \in W_h$. Obviously,

$$\begin{aligned} \|\bar{w}_h - w_h\|_1^h &\le C\big(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + t^2 \|\, \mathbf{grad}_h(r - r_h)\|_0 \big) \\ &\le C\big(h + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \big), \end{aligned}$$

which implies

$$\|w - w_h\|_{1,\Omega}^h \le \|w - \bar{w}_h\|_1^h + \|\bar{w}_h - w_h\|_1^h \le C(h + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0) \le C_\epsilon h t^{-\epsilon}.$$

This completes (4.7.10).

We will use the duality argument again to estimate $\|p - p_h\|_{-2}$. To do so, we introduce the following auxiliary problem.

Find $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^1 \times \hat{H}^1$ such that

$$(\mathcal{C}\mathcal{E}(\boldsymbol{\psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\boldsymbol{\psi}, \mathbf{curl}\, P) = 0 \qquad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}^1, \quad (4.7.29)$$

$$-(\mathbf{curl}\, q, \boldsymbol{\Phi}) - \lambda^{-1} t^2 (\mathbf{curl}\, q, \mathbf{curl}\, P) = (K, q) \quad \text{for all } q \in H^1, \quad (4.7.30)$$

for any $K \in \hat{L}^2$. By Corollary 4.2.4, this problem admits a unique solution, and that

$$\|\boldsymbol{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon} \|K\|_2. \quad (4.7.31)$$

By definition,

$$\|p - p_h\|_{-2} = \sup_{\substack{K \in \mathring{H}^2 \\ K \neq 0}} \frac{(p - p_h, K)}{\|K\|_2}. \tag{4.7.32}$$

In (4.7.30), we take q to be $p - p_h$ and apply (4.2.4), (4.2.5), (4.4.5), (4.4.6), and (4.7.29) to obtain

$$\begin{aligned}(p - p_h, K) = &- \big(\, \mathbf{\Phi} - \mathbf{\Pi\Phi}, \mathbf{curl}(p - p_h) \,\big) - \lambda^{-1} t^2 \big(\, \mathbf{curl}(p - p_h), \mathbf{curl}(P - \pi_h P) \,\big) \\ &- \big(\, \mathcal{C}\mathcal{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathcal{E}(\mathbf{\Phi} - \mathbf{\Pi\Phi}) \,\big) + \big(\, \boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{curl}(\pi_h P - P) \,\big) \\ &+ \big(\, \mathbf{grad}_h (r - r_h), \mathbf{\Pi\Phi} \,\big) \end{aligned} \tag{4.7.33}$$

Using the Schwartz inequality, Theorem 4.7.1, and (4.7.31) in (4.7.33), we get

$$|(p - p_h, K)| \le C_\epsilon h t^{-\epsilon} \big(\, \|\mathbf{\Phi}\|_{3/2} + t\|P\|_{3/2} + \|P\|_{1/2+\epsilon} \,\big). \tag{4.7.34}$$

Combining (4.7.31)–(4.7.34), we arrive at

$$\|p - p_h\|_{-2} \le C_\epsilon h t^{-2\epsilon}.$$

Similarly, we can prove

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{-1} \le C_\epsilon h t^{-2\epsilon}.$$

Since ϵ is an arbitrary number, then (4.7.12) is proved. □

Equipped with the above result, we are able to prove the following interior estimate for the Arnold-Falk element for the rotation $\boldsymbol{\phi}$.

Theorem 4.7.4. *Let Ω be a convex polygon and Ω_0, $\Subset \Omega$ an interior domain. Let g be a smooth function. Assume that $\mathcal{T}_h$ is quasi-uniform. Suppose that $(w, \boldsymbol{\phi})$ solves (4.2.1) and $(w_h, \boldsymbol{\phi}_h)$ solves (4.4.1). Then there exists a number $h_1 \ge 0$, such that for all $h \in (0, h_1]$,*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,\Omega_0} \le C_\epsilon\, h t^{-\epsilon}, \tag{4.7.35}$$

where C_ϵ is independent of t and h.

Proof. First choose Ω_1 such that $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Then note that $\|\cdot\|_{s,\Omega_1} \leq \|\cdot\|_{s,\Omega}$. Combining Theorem 4.7.3 with Theorem 4.6.3 with $\alpha = 2$, $\beta = 2$, (4.7.35) can be obtained. □

Because the Brezzi-Fortin element (cf. [15]) is also based on the variational formulation (4.2.3)-(4.2.6), we have the following result.

Corollary 4.7.5. *Assume that the Brezzi-Fortin method [15] is used to solve* (4.2.3)-(4.2.6). *Then, under the same conditions of Theorem 4.7.4, Theorem 4.7.3 and Theorem 4.7.4 hold.*

4.8 Numerical Results

In this section we give the results of computations of the solutions to the Arnold-Falk element for the Reissner-Mindlin plate model. Through a model problem, we show that the Arnold-Falk approximation for rotation ϕ does not achieve the global first order convergence rate in the energy norm for the soft simply supported plate, but it does have first order convergence rate for the transverse displacement w. We will also show that the Arnold-Falk method obtains the first order convergence rate for the rotation in the region away from the boundary layer. Thereafter, numerical computations conform to the theoretical predictions.

We will take the domain Ω to be the unit square. Since we know the exact solution of the semi-infinite ($y > 0$) Reissner-Mindlin plate when the load function $g(x, y) = \cos(x)$ and the plate is soft simply supported on the boundary $y = 0$, we can simply restrict this solution to Ω. By doing so, we obtain the exact solution of the unit square plate with the hard clamped boundary condition on the left, upper, and right edges, and soft simply supported boundary condition on the lower edge

$(0 < x < 1,\ y = 0)$ (cf. [8]). And the lower edge $(0 < x < 1,\ y = 0)$ is where the boundary layer occurs.

We take $E = 1$, $\nu = 3/10$, and $\kappa = 5/6$. Moreover, the mesh is taken to be uniform. The interior domain is taken to be the upper half of the unit square (since the boundary layer only exists near the lower edge $0 < x < 1$, $y = 0$). All computations were performed on a Sun SPARCStation 2 using the Modulef (INRIA) package.

A distinguished feature of this test problem is that the exact solution has the following property: $\phi_1 \in H^{3/2}(\Omega)$ and $\phi_2 \in H^{5/2}(\Omega)$, i.e., ϕ_1 has a stronger boundary layer than ϕ_2 does (cf. [5]). The numerical results unmistakenly express this difference.

In each graph (of Figure 4.1–Figure 4.6) the H^1 norms of the errors on the global domain and the interior domain, are plotted as functions of the mesh size h. The values of h are 1/2, 1/4, 1/6, 1/8, 1/10, 1/12, 1/16, 1/20, and 1/24. Both axes have been transformed logarithmically so that the slope of the error curves gives the apparent rate of convergence as h tends to zero. Absolute errors are shown.

Figures 4.1–4.2 show the approximation errors in H^1 norm of ϕ_1 and ϕ_2 for $t = 1$ and the first order optimal convergence rate is as expected. And there is no difference between the rate on the whole domain and that on the upper half unit square. Figures 4.3–4.4 show the errors in H^1 norm of ϕ_1 and ϕ_2 for $t = 0.0001$. It is clear that when t is small, the boundary layer effect of ϕ_1 comes into play and as a result, we only see a 1/2 order convergence rate for $\|\phi_1 - \phi_1^h\|_1$ on the whole domain. However, away from the boundary layer, the optimal first order convergence rate is recovered.

Figures 4.5–4.6 show the errors in H^1 norm of the transverse displacement w for $t = 1$ and 0.0001. In all cases, the first order convergence rate is observed, because there is no boundary layer in the transverse displacement.

Figures 4.7–4.12 show the errors in L^2 norm for variables ϕ_1, ϕ_2, and w, with the thickness of the plate $t = 1$ and $t = 0.0001$, respectively. We note that in the interior domain, the optimal convergence rate (second order) is observed, but this cannot be proved by the current method. (Though we did not explicitly state a theorem about the interior estimate in the L^2 norm in section 4.6, it is not difficult to do so in the light of Chapter 2 and Chapter 3.) The global convergence rates in the L^2 norm are also higher than we actually proved in Section 4.7. We do not know at the moment whether they are of the special feature of the test problem or they simply indicate that the convergence analysis can be improved.

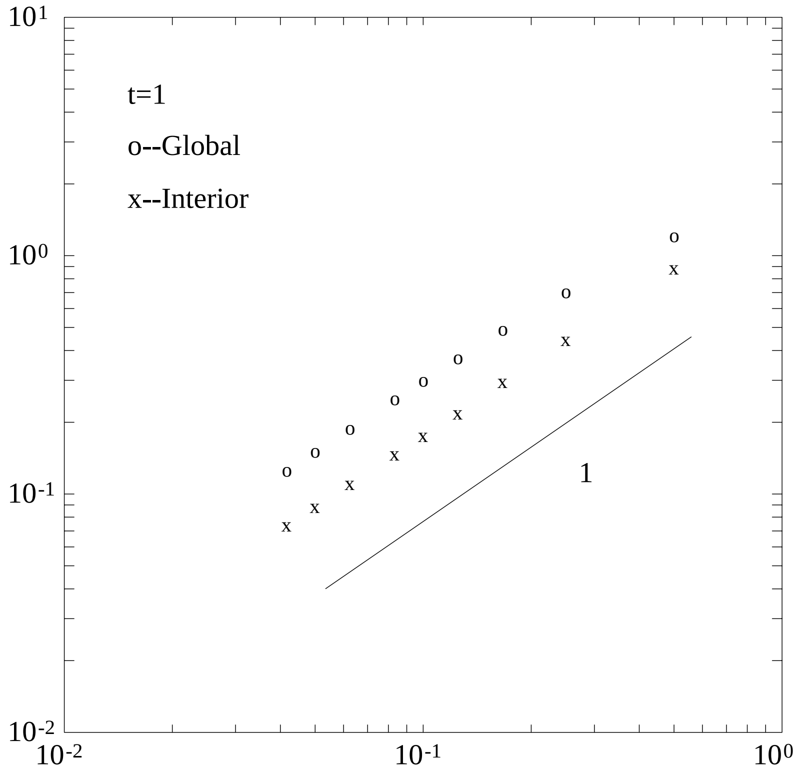

Figure 4.1: Errors in H^1 Norm for ϕ_1

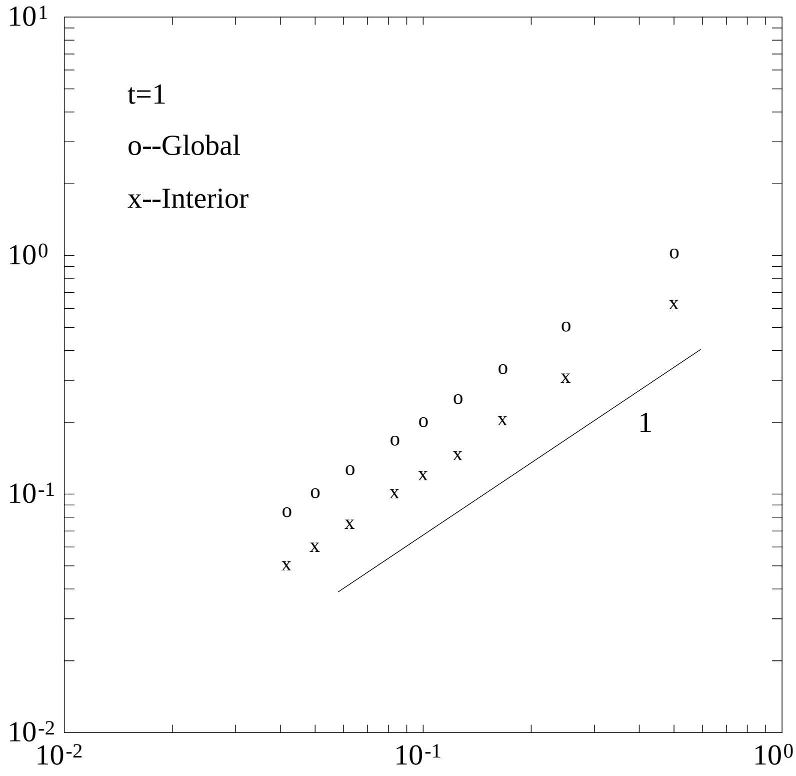

Figure 4.2: Errors in H^1 Norm for ϕ_2

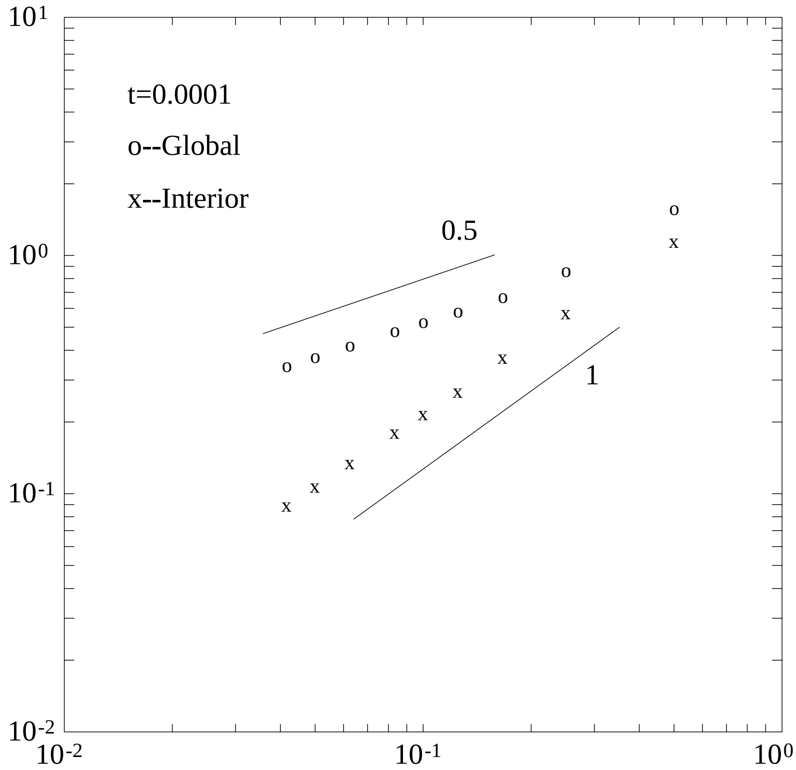

Figure 4.3: Errors in H^1 Norm for ϕ_1

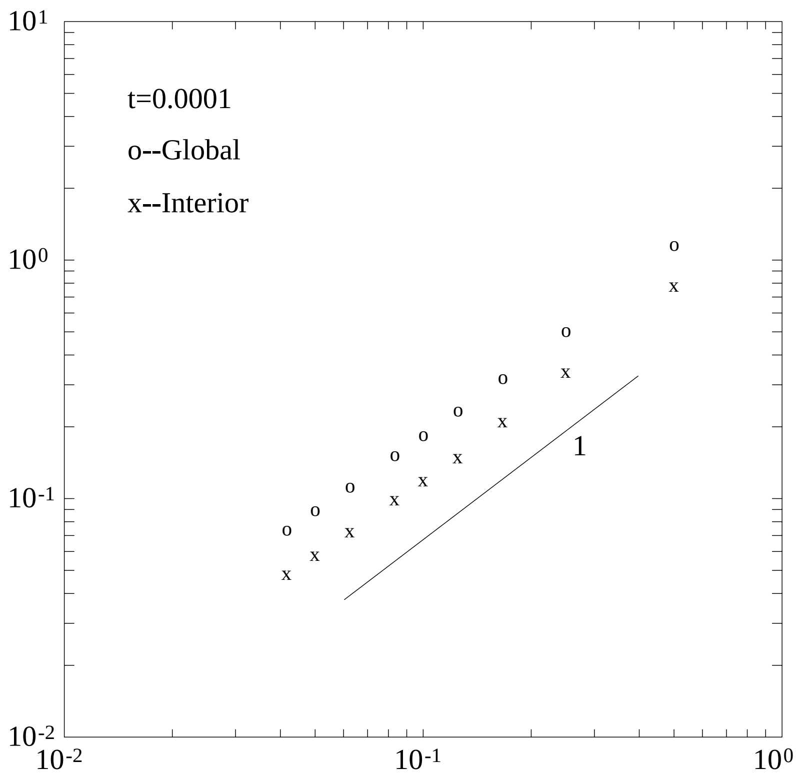

Figure 4.4: Errors in H^1 Norm for ϕ_2

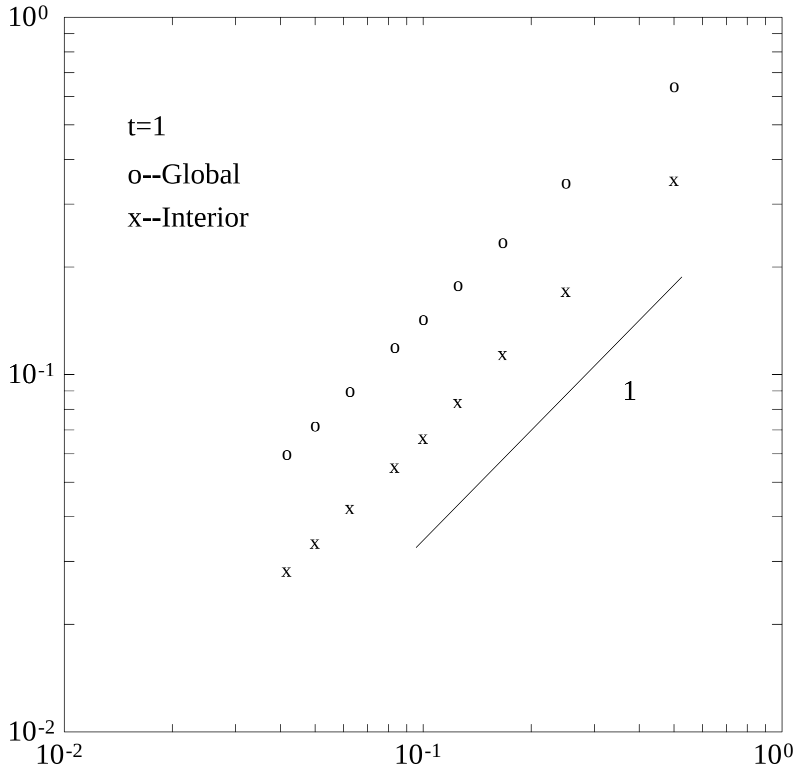

Figure 4.5: Errors in H^1 Norm for w

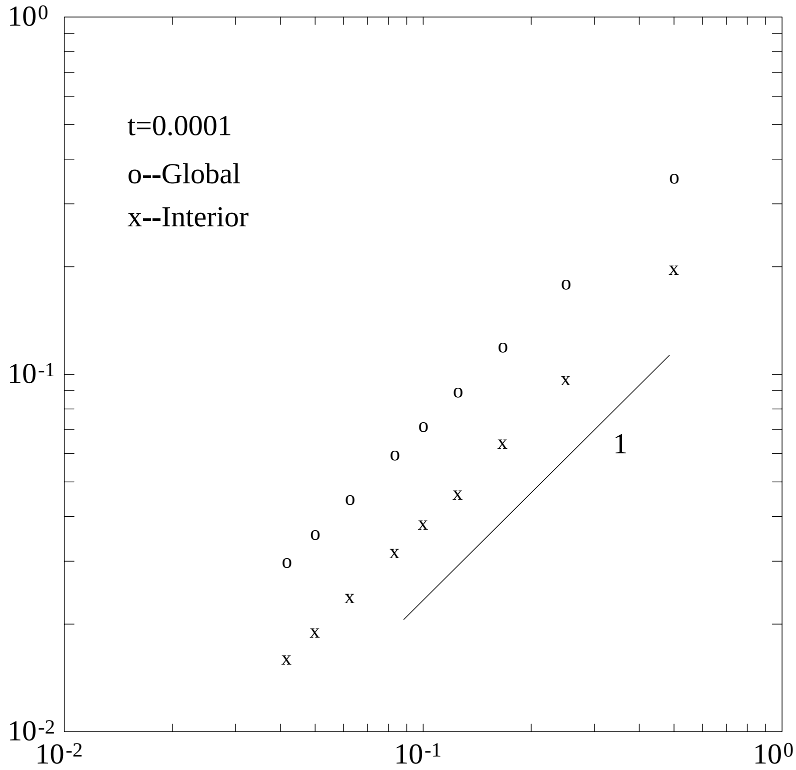

Figure 4.6: Errors in H^1 Norm for w

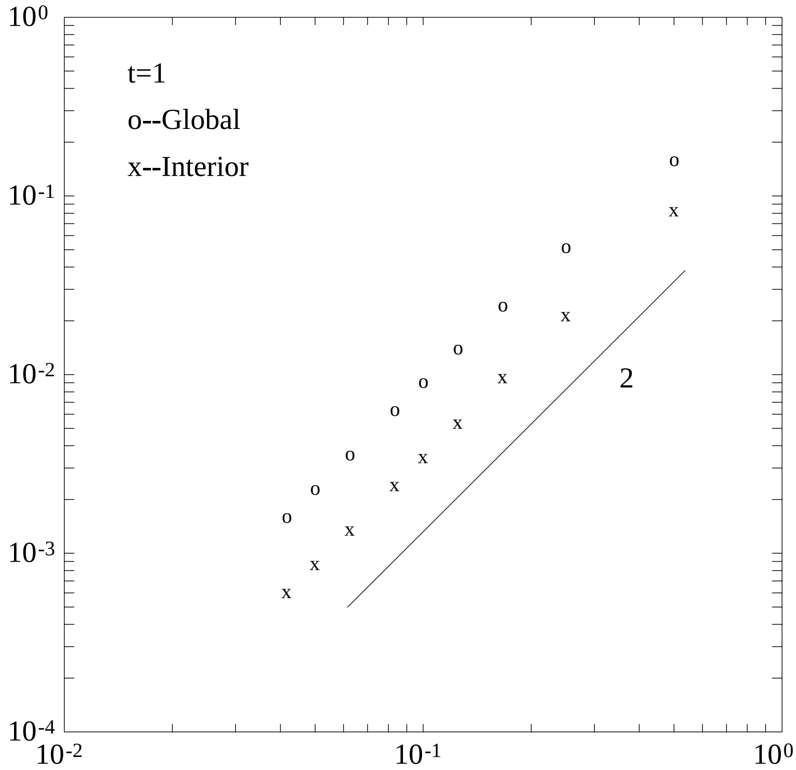

Figure 4.7: Errors in L^2 Norm for ϕ_1

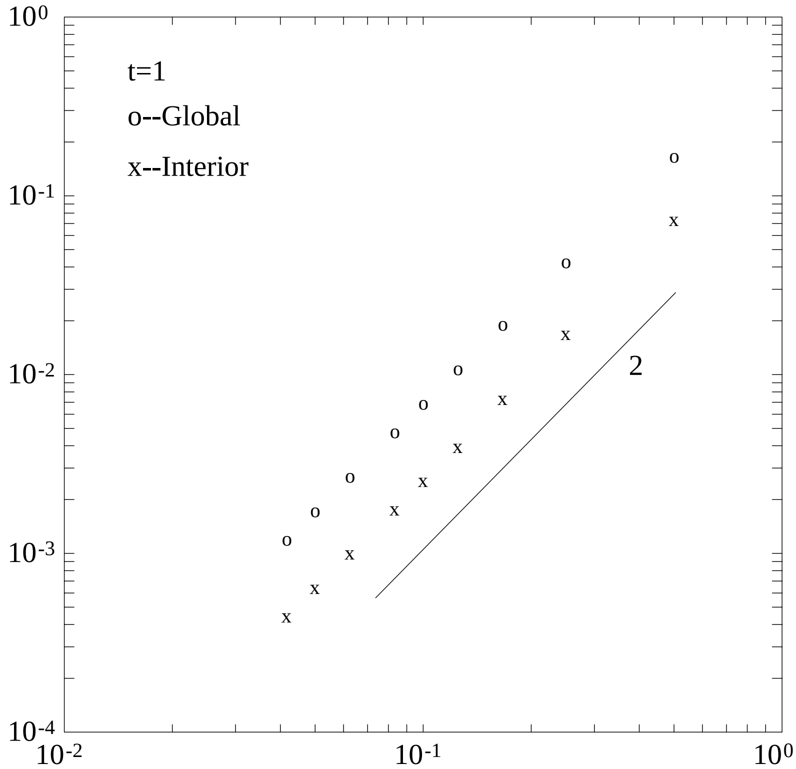

Figure 4.8: Errors in L^2 Norm for ϕ_2

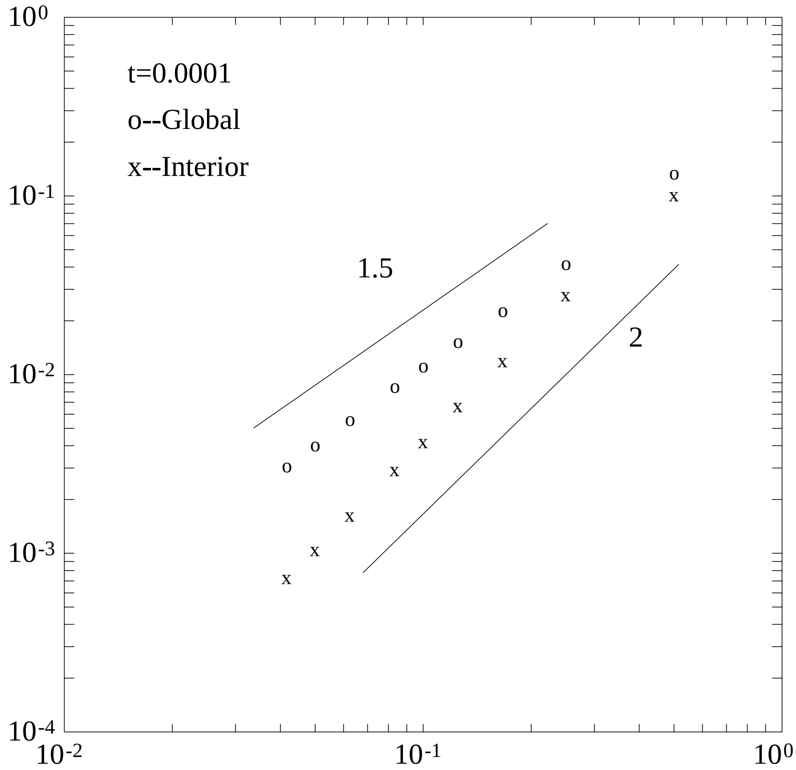

Figure 4.9: Errors in L^2 Norm for ϕ_1

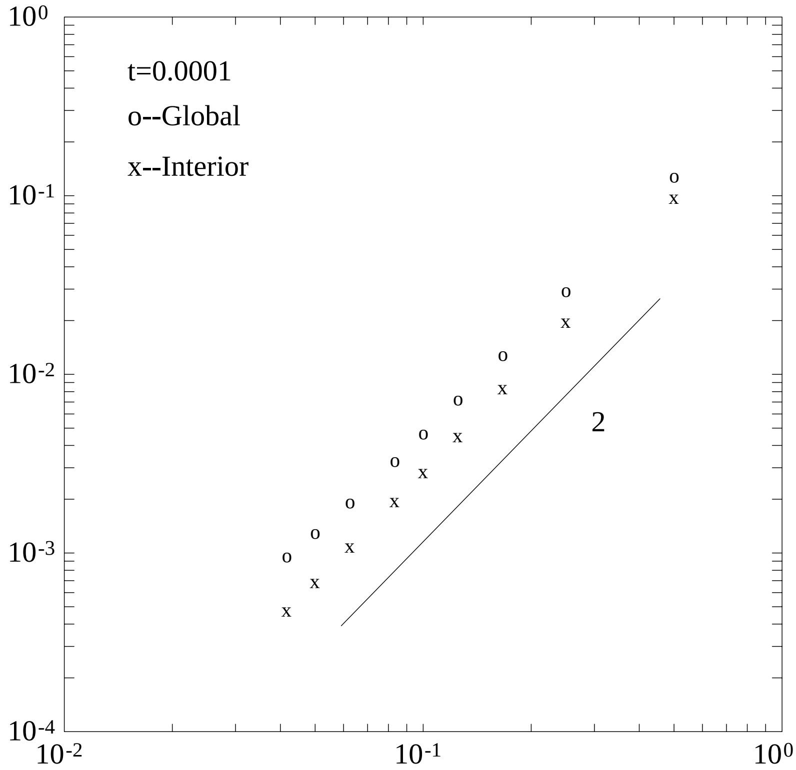

Figure 4.10: Errors in L^2 Norm for ϕ_2

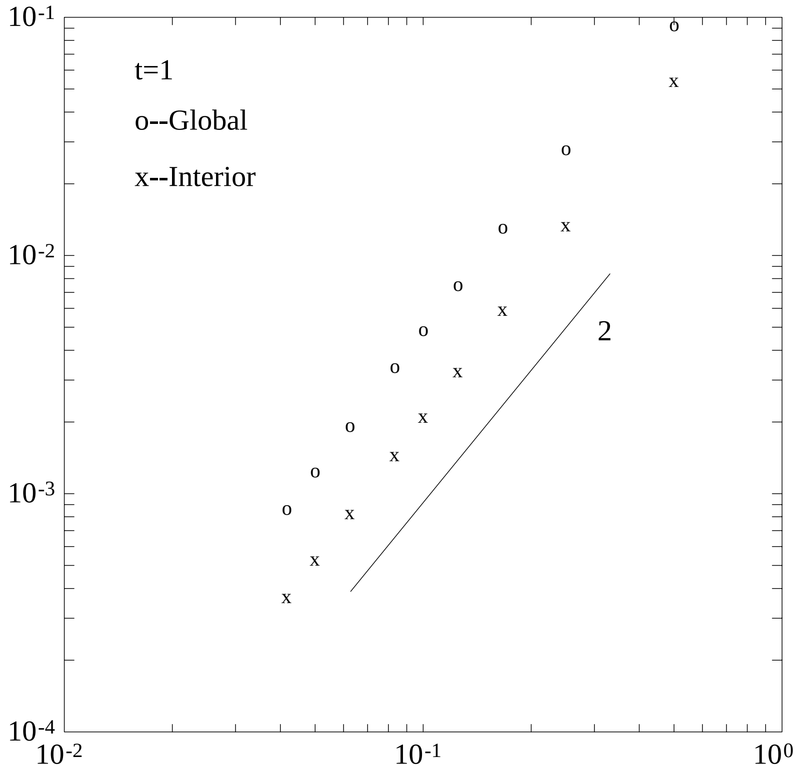

Figure 4.11: Errors in L^2 Norm for w

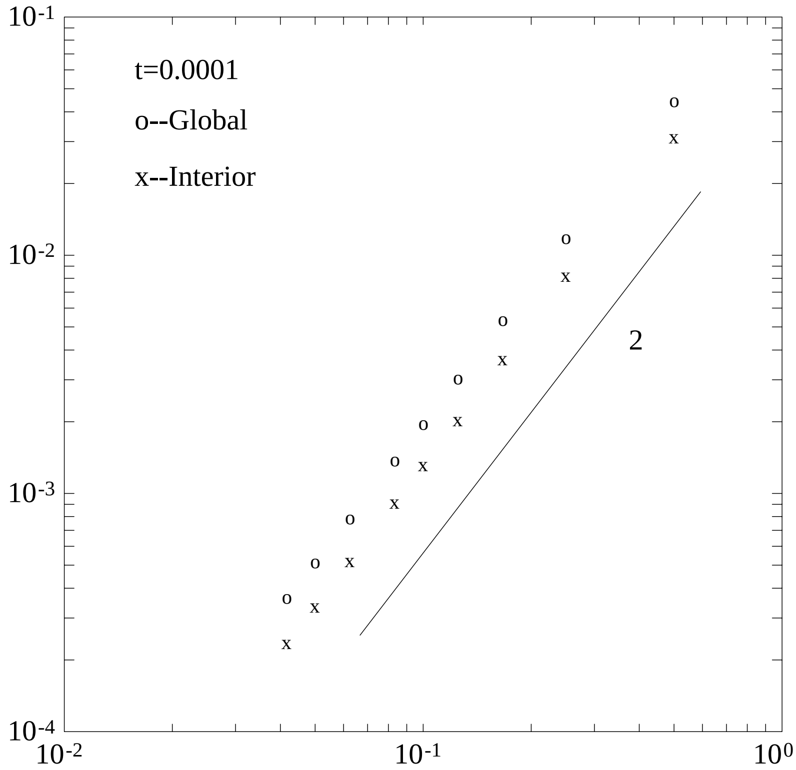

Figure 4.12: Errors in L^2 Norm for w

APPENDIX A

AN APPROXIMATION RESULT

5.1 Introduction

The purpose of this appendix is to prove Theorem 5.3.1, which is due to Arnold [7]. This approximation result was used extensively in Chapter 4 (as Theorem 4.7.1) for proving the global convergence of the Arnold-Falk element for the Reissner-Mindlin plate model under the simply supported boundary condition.

Recall that P_h is the space of continuous piecewise linear functions. We shall start with a result by Scott and Zhang [39].

Theorem 5.1.1. *Assume that Ω is a convex polygon. Let $\Gamma = \partial\Omega$ and $P_h^\Gamma = \{ v|_\Gamma : v \in P_h \} \subset H^1(\Gamma)$. There exists a projection $I_h : H^1 \to P_h$ such that if $u|_\Gamma \in P_h^\Gamma$ then $I_h u|_\Gamma = u|_\Gamma$. Moreover*

$$\|u - I_h u\|_s \le C h^{l-s} \|u\|_l, \quad \textit{for} \quad 0 \le s \le l \le 2, \; l > 1/2. \tag{5.1.1}$$

Using this, we can quickly prove:

Lemma 5.1.2. *Let $w \in H^1$ be a function for which $w|_\Gamma \in P_h^\Gamma$. Define $w_h \in P_h$ by*

$$\int_\Omega \mathbf{grad}\, w_h \cdot \mathbf{grad}\, v = \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, v \quad \textit{for all } v \in \mathring{P}_h, \quad w_h = w \quad \textit{on } \Gamma.$$

Then

$$\|w - w_h\|_1 \le C \inf_{\substack{\chi \in P_h \\ \chi = w \;\text{ on } \Gamma}} \|w - \chi\|_1, \tag{5.1.2}$$

$$\|w - w_h\|_0 \le C h \|w - w_h\|_1, \tag{5.1.3}$$

$$\|w - w_h\|_s \le C h^{t-s} \|w\|_t, \quad s = 0, 1, \quad t = 1, 2. \tag{5.1.4}$$

Proof. The first two estimates are completely standard. We take χ to be the interpolant of Theorem 5.1.1 to get the third. □

The outline of this chapter is as follows. Section 5.2 constructs the approximation operator and section 5.3 proves that it has the desired property.

5.2 The Construction of the Approximation Operator

In this section we study a finite element method for the nonhomogeneous Dirichlet problem for the Poisson equation. We will prove some of the properties of the finite element method here and we will show in the next section that the approximation operator determined by the finite element solution is the one we need.

For simplicity, we will use notation $|\cdot|_t$ to denote $\|\cdot\|_{t,\partial\Omega}$ in this chapter (instead of its old meaning as the semi-norm on H^t).

Lemma 5.2.1. *Given $p \in H^1$, let $g = p|_\Gamma$ and let g_h be the $L^2(\Gamma)$-projection of g into P_h^Γ. Define $p_h \in P_h$ by*

$$\int_\Omega \mathbf{grad}\, p_h \cdot \mathbf{grad}\, q = \int_\Omega \mathbf{grad}\, p \cdot \mathbf{grad}\, q \quad \text{for all } q \in \mathring{P}_h, \quad p_h = g_h \quad \text{on } \Gamma. \tag{5.2.1}$$

Then

$$\|p - p_h\|_s \le C h^{t-s} \|p\|_t, \quad 0 \le s \le 1, \quad 1 \le t \le 2.$$

Proof. Define $\bar{p}_h \in H^1$ by

$$\Delta \bar{p}_h = \Delta p \quad \text{in } \Omega, \quad \bar{p}_h = g_h \quad \text{on } \Gamma.$$

Since $p - \bar{p}_h$ is harmonic, we have

$$\|p - \bar{p}_h\|_0 \le C|g - g_h|_{-1/2}, \qquad \|p - \bar{p}_h\|_1 \le C|g - g_h|_{1/2}. \tag{5.2.2}$$

Now using a standard duality argument and standard approximation results for the L^2-projection into P_h^Γ together with the trace theorem we get

$$\begin{aligned} |g - g_h|_{-1/2} &= \sup_{f \in H^{1/2}(\Gamma)} \frac{\langle g - g_h, f \rangle}{|f|_{1/2}} \\ &= \sup_{f \in H^{1/2}(\Gamma)} \frac{\langle g - g_h, f - f^I \rangle}{|f|_{1/2}} \\ &\le C h^{1/2} |g - g_h|_0 \le C h |g|_{1/2} \le C h \|p\|_1, \end{aligned} \tag{5.2.3}$$

where f^I is the $L^2(\Gamma)$ projection of f on P_h^Γ. Although Γ is not sufficiently smooth to define the space $H^{3/2}(\Gamma)$ intrinsically, we can define $\bar{H}^{3/2}(\Gamma)$ to be the space of functions in $H^1(\Gamma)$ whose restrictions to each edge e of the polygon belong to $H^{3/2}(e)$, and use as the norm

$$v_{3/2} := \left(\sum_{e \in \partial\Omega} \|v\|^2_{H^{3/2}(e)} \right)^{1/2} .$$

Then

$$|g - g_h|_{-1/2} \le C h^{1/2} |g - g_h|_0 \le C h^2 |g|_{3/2} \le C h^2 \|p\|_2. \tag{5.2.4}$$

From $|g - g_h|_0 \le C|g|_0$ and the inverse inequality we can obtain $|g - g_h|_1 \le C|g|_1$. Then, by the interpolation theorem we get

$$|g - g_h|_{1/2} \le C|g|_{1/2} \le C\|p\|_1, \quad |g - g_h|_{1/2} \le C h |g|_{3/2} \le C h \|p\|_2. \tag{5.2.5}$$

Combining (5.2.2)–(5.2.5) we get

$$\|p - \bar{p}_h\|_s \le C h^{t-s} \|p\|_t, \quad s = 0, 1, \quad t = 1, 2. \tag{5.2.6}$$

Now

$$\int_\Omega \mathbf{grad}\, p_h \cdot \mathbf{grad}\, q = \int_\Omega \mathbf{grad}\, \bar{p}_h \cdot \mathbf{grad}\, q \quad \text{for all } q \in \mathring{P}_h, \quad p_h = \bar{p}_h \quad \text{on } \Gamma,$$

Then, using (5.1.4) in the case $t = 1$ we obtain

$$\|\bar{p}_h - p_h\|_s \le Ch^{1-s}\|\bar{p}_h\|_1, \quad s = 0, 1.$$

Thus combining the above with (5.2.6) in the case $t = 1$ we get

$$\|p - p_h\|_s \le Ch^{1-s}(\|p\|_1 + \|\bar{p}_h\|_1) \le Ch^{1-s}\|p\|_1, \quad s = 0,\ 1,$$

where in the last step we use (5.2.6) for $s = 1$ and $t = 1$.

Now let $I_h p$ be the usual piecewise linear interpolant of p so that $\widetilde{g}_h := I_h p|_\Gamma$ is the piecewise linear interpolant of g, and define $\widetilde{p}_h \in P_h$ by

$$\int_\Omega \mathbf{grad}\, \widetilde{p}_h \cdot \mathbf{grad}\, q = \int_\Omega \mathbf{grad}\, p \cdot \mathbf{grad}\, q \quad \text{for all } q \in \mathring{P}_h, \quad \widetilde{p}_h = \widetilde{g}_h \quad \text{on } \Gamma.$$

Then

$$\|p - \widetilde{p}_h\|_1 \le C \inf_{\substack{\chi \in P_h \\ \chi = \widetilde{g}_h \quad \text{on } \Gamma}} \|p - \chi\|_1 \le C\|p - I_h p\|_1 \le Ch\|p\|_2. \tag{5.2.7}$$

Next, define $w \in H^1$ by

$$\Delta\, w = 0 \quad \text{in } \Omega, \quad w = \widetilde{g}_h - g_h \quad \text{on } \Gamma.$$

Note that $\widetilde{p}_h - p_h \in P_h$, and

$$\int_\Omega \mathbf{grad}(\widetilde{p}_h - p_h) \cdot \mathbf{grad}\, q = \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, q = 0 \quad \text{for all } q \in \mathring{P}_h,$$
$$\widetilde{p}_h - p_h = w \quad \text{on } \Gamma.$$

Then by the Lemma 5.1.2, we have $\|\widetilde{p}_h - p_h\|_s \le Ch^{1-s}\|w\|_1$, for $s = 0,\ 1$. Since

$$\|w\|_1 \le C|\widetilde{g}_h - g_h|_{1/2} \le C(|g - \widetilde{g}_h|_{1/2} + |g - g_h|_{1/2}) \le Ch|g|_{3/2} \le Ch\|p\|_2,$$

we get

$$\|\widetilde{p}_h - p_h\|_s \le Ch^{2-s}\|p\|_2,$$

which, together with (5.2.7) gives $\|p - p_h\|_1 \le Ch\|p\|_2$.

Finally we use duality to prove that $\|p-p_h\|_0 \le Ch\|p-p_h\|_1$, and thus $\|p-p_h\|_0 \le Ch\|p\|_2$. Namely, we define z by

$$-\Delta z = p - p_h \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma.$$

Then $\|z\|_2 \le C\|p - p_h\|_0$, and

$$\begin{aligned}
\|p - p_h\|_0^2 &= -\int_\Omega (p - p_h)\,\Delta z = \int_\Omega \mathbf{grad}(p - p_h)\cdot \mathbf{grad}\, z - \int_\Gamma (g - g_h)\frac{\partial z}{\partial n_T} \\
&\le \|p - p_h\|_1 \inf_{\chi \in \mathring{P}_h} \|z - \chi\|_1 + |g - g_h|_{-1/2}\left|\frac{\partial z}{\partial n_T}\right|_{1/2} \\
&\le Ch\|p - p_h\|_1\|z\|_2 + Ch^2|g|_{3/2}\|z\|_2 \\
&\le Ch^2\|p\|\|p - p_h\|_0,
\end{aligned}$$

as desired.

This completes the proof for $s = 0$ and 1, and $t = 1$ and 2. The extension to real indices follows by interpolation. □

5.3 The Main Result

For $p \in H^1$, let $\pi_h p = p_h$ be the finite element solution defined in Lemma 5.2.1. In the following, we shall prove that π_h is what we need. To do so, we should keep in mind two important properties of π_h: equation (5.2.1) and that π_h preserves P_h^Γ on the boundary.

Theorem 5.3.1. *Assume that $u \in H^2$, where Ω is a convex polygon. Then the operator $\pi_h : H^1 \to P_h$ constructed in Lemma 5.2.1 satisfies*

$$\|p - \pi_h p\|_0 \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}, \tag{5.3.1}$$

$$\|p - \pi_h p\|_1 \le C h^{1/2} \|p\|_{3/2}, \tag{5.3.2}$$

$$|p - \pi_h p|_{-1/2} \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}, \tag{5.3.3}$$

for any $0 < \epsilon \le 1/2$. Here C is independent of ϵ and C_ϵ depends on ϵ, but not h.

Proof. Inequality (5.3.2) is already proved in Lemma 5.2.1. Inequality (5.3.3) is also straightforward: since $\pi_h p$ is the $L^2(\Gamma)$ projection of p on P_h^Γ, we have

$$|p - \pi_h p|_{-1/2} \le C h^{1/2} |p - \pi_h p|_0 \le C_\epsilon h^{1/2+\epsilon} |p|_\epsilon \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon},$$

where we use the trace theorem in the last step.

We now prove (5.3.1) in three steps: first for $p \in \mathring{H}^1$, then for p such that $p|_\Gamma \in P_h^\Gamma$, and finally for all $p \in H^1$.

Using an inverse inequality [43,Theorem 3.1], we obtain

$$\|z_h\|_{3/2-\epsilon} \le C_\epsilon h^{-1/2+\epsilon} \|z_h\|_1 \quad \text{for all } z_h \in P_h,$$

which implies that $z_h \in H^{3/2-\epsilon}$. For all $T \in \mathcal{T}_h$, applying inequality [32]

$$\|u\|_{1+s,T} \le C(\delta \|u\|_{2,T} + \delta^{-s/(1-s)} \|u\|_{1,T})$$

for $u = z - \pi_h z$, $\delta = h^{1/2-\epsilon}$, and $s = 1/2 - \epsilon$ yields

$$\|z - \pi_h z\|_{3/2-\epsilon,T} \le C_\epsilon (h^{1/2-\epsilon} \|z - \pi_h z\|_{2,T} + h^{-1/2-\epsilon} \|z - \pi_h z\|_{1,T}) \quad \text{for all } z \in H^2.$$

Summing up inequalities of above type for all $T \in \Omega$ and noting that the second order derivative of $\pi_h z$ vanishes, we obtain

$$\|z - \pi_h z\|_{3/2-\epsilon} \le C_\epsilon (h^{1/2-\epsilon} \|z\|_2 + h^{-1/2-\epsilon} \|z - \pi_h z\|_1) \quad \text{for all } z \in H^2).$$

Then applying Lemma 5.2.1 for $s = 1$ and $t = 2$ yields

$$\|z - \pi_h z\|_{3/2-\epsilon} \le C_\epsilon h^{1/2-\epsilon} \|z\|_2 \quad \text{for all } z \in H^2. \tag{5.3.4}$$

Now if $p \in \mathring{H}^1$ then $\pi_h p \in \mathring{P}_h$, so if both $p,\, z \in \mathring{H}^1$,

$$(\mathbf{grad}\, \pi_h p, \mathbf{grad}\, z) = (\mathbf{grad}\, \pi_h p, \mathbf{grad}\, \pi_h z) = (\mathbf{grad}\, p, \mathbf{grad}\, \pi_h z).$$

For a given $p \in \mathring{H}^1$, we will use a duality argument to get (5.3.1). Taking $z \in H^2 \cap \mathring{H}^1$ with $-\Delta z = p - \pi_h p$ and $\|z\|_2 \le C\|p - \pi_h p\|_0$ for $\pi_h z$ as described in Lemma 5.2.1, we get

$$\begin{aligned}
\|p - \pi_h p\|_0^2 &= (\mathbf{grad}(p - \pi_h p), \mathbf{grad}\, z) = (\mathbf{grad}\, p, \mathbf{grad}(z - \pi_h z)) \\
&\le \|\,\mathbf{grad}\, p\|_{\epsilon-1/2} \|\,\mathbf{grad}(z - \pi_h z)\|_{1/2-\epsilon} \le \|p\|_{1/2+\epsilon} \|z - \pi_h z\|_{3/2-\epsilon} \\
&\le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon} \|z\|_2 \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon} \|p - \pi_h p\|_0,
\end{aligned}$$

which proves (5.3.1) for $p \in \mathring{H}^1$.

Assume $p \in H^1$ has the property that $p|_\Gamma \in P_h^\Gamma$. Let I_h denote the Scott-Zhang interpolant [39]. Then, since $I_h p = p$ on Γ and, using what we just proved and the fact that I_h is bounded in $H^{1/2+\epsilon}$, we obtain

$$\begin{aligned}
\|p - \pi_h p\|_0 &= \|(p - I_h p) - \pi_h (p - I_h p)\|_0 \\
&\le C_\epsilon h^{1/2+\epsilon} \|p - I_h p\|_{1/2+\epsilon} \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}.
\end{aligned}$$

This completes the proof of the second case. Finally for the general case of $p \in H^1$ we use the same decomposition as in the proof of Lemma 5.2.1. Namely we define $\bar{p}_h \in H^1$ by

$$\Delta\, \bar{p}_h = \Delta\, p \quad \text{in } \Omega, \quad \bar{p}_h = g_h \quad \text{on } \partial\Omega,$$

where g_h is the $L^2(\Gamma)$-projection of $g = p|_\Gamma$ into P_h^Γ. Then

$$\|p - \bar{p}_h\|_0 \le |g - g_h|_{-1/2} \le C_\epsilon h^{1/2+\epsilon} |g|_\epsilon \le C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}.$$

Also

$$\|p - \bar{p}_h\|_{1/2+\epsilon} \leq |g - g_h|_\epsilon \leq C_\epsilon |g|_\epsilon \leq C_\epsilon \|p\|_{1/2+\epsilon}.$$

Here we have used an inverse inequality to obtain that the $L^2(\partial\Omega)$-projection is bounded in $H^\epsilon(\partial\Omega)$. We thus have

$$\|\bar{p}_h\|_{1/2+\epsilon} \leq C_\epsilon \|p\|_{1/2+\epsilon}.$$

Finally we have $\pi_h \bar{p}_h = \pi_h p$, so

$$\|\bar{p}_h - \pi_h p\|_0 = \|\bar{p}_h - \pi_h \bar{p}_h\|_0 \leq C_\epsilon h^{1/2+\epsilon} \|\bar{p}_h\|_{1/2+\epsilon} \leq C_\epsilon h^{1/2+\epsilon} \|p\|_{1/2+\epsilon}.$$

This completes the proof. $\square$

APPENDIX B

A REGULARITY RESULT

The purpose of this Appendix is to prove Theorem 6.1 on the regularity of the exact solution of the singularly perturbed Stokes-like system under the soft simply supported boundary condition. This is done assuming that the domain Ω is smooth. So far, we cannot prove the same result for a convex polygonal domain.

Theorem 6.1. *Let Ω denote a smooth domain, and let $\boldsymbol{F} \in \boldsymbol{H}^1$ and $K \in H^2 \cap \hat{L}^2$. Then there exists a unique solution $(\boldsymbol{\Phi}, P) \in \boldsymbol{H}^2 \times H^1 \cap \hat{L}^2$ to the equations*

$$(C\mathcal{E}(\boldsymbol{\Psi}), \mathcal{E}(\boldsymbol{\Phi})) - (\boldsymbol{\Psi}, \mathbf{curl}\, P) = (\boldsymbol{F}, \boldsymbol{\Psi}) \quad \text{for all } \boldsymbol{\Psi} \in \boldsymbol{H}^1, \tag{6.1}$$

$$-(\boldsymbol{\Phi}, \mathbf{curl}\, Q) - \lambda^{-1} t^2 (\mathbf{curl}\, Q, \mathbf{curl}\, P) = (K, Q) \quad \text{for all } Q \in H^1. \tag{6.2}$$

Moreover,

$$\|\boldsymbol{\Phi}\|_2 + \|P\|_1 + t\|P\|_2 \le C\big(t^{-1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t(\|\boldsymbol{F}\|_1 + \|K\|_2)\big), \tag{6.3}$$

$$\|P\|_{1/2} \le C\big(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2} + t(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\big), \tag{6.4}$$

$$\|\boldsymbol{\Phi}\|_{3/2} + t\|P\|_{3/2} \le C\big(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2} + t^{3/2}(\|\boldsymbol{F}\|_1 + \|K\|_2)\big). \tag{6.5}$$

Proof. We first define some notations. Let

$$M_{\boldsymbol{n}}\boldsymbol{\phi} := \boldsymbol{n} \cdot C\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = D\left(\frac{\partial \boldsymbol{\phi}}{\partial n} \cdot \boldsymbol{n} + \nu \frac{\partial \boldsymbol{\phi}}{\partial s} \cdot \boldsymbol{s}\right),$$

$$M_{\boldsymbol{s}}\boldsymbol{\phi} := \boldsymbol{s} \cdot C\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = \frac{D(1-\nu)}{2}\left(\frac{\partial \boldsymbol{\phi}}{\partial n} \cdot \boldsymbol{s} + \nu \frac{\partial \boldsymbol{\phi}}{\partial n} \cdot \boldsymbol{n}\right),$$

on $\partial\Omega$, where $\boldsymbol{s}$ and $\boldsymbol{n}$ are the unit tangential and outward normal directions, respectively. Then consider a reduced problem:

Find $(\boldsymbol{\Phi}_0, P_0) \in \boldsymbol{H}^2 \times H^1 \cap \hat{L}^2$ such that

$$-\operatorname{\mathbf{div}} \mathcal{C}\mathcal{E}(\boldsymbol{\Phi}_0) - \operatorname{\mathbf{curl}} P_0 = \boldsymbol{F}, \tag{6.6}$$

$$-\operatorname{rot} \boldsymbol{\Phi}_0 = K, \tag{6.7}$$

together with boundary conditions

$$M_{\boldsymbol{n}} \boldsymbol{\Phi}_0 = 0, \qquad \boldsymbol{\Phi}_0 \cdot \boldsymbol{s} = 0.$$

By the standard theory on the elliptic system, we have

$$\|\boldsymbol{\Phi}_0\|_{s+1} + \|P_0\|_s \leq C(\|\boldsymbol{F}\|_{s-1} + \|K\|_s) \quad \text{for all real } s \geq 0. \tag{6.8}$$

Now set

$$\boldsymbol{\Phi}^E = \boldsymbol{\Phi} - \boldsymbol{\Phi}_0, \qquad P^E = P - P_0.$$

In the light of (6.8), we need only estimate $\boldsymbol{\Phi}^E$ and P^E. Actually, we have the following theorem.

Theorem 6.2. *Under the same conditions of Theorem 6.1, there exists a constant C depending only on the domain Ω such that*

$$\begin{aligned} &\|\boldsymbol{\Phi}^E\|_1 + \|P^E\|_0 + t\|P^E\|_1 \\ &\leq C\big(t^{1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t^{3/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\big), \end{aligned} \tag{6.9}$$

$$\begin{aligned} &\|\boldsymbol{\Phi}^E\|_2 + t\|P^E\|_2 \\ &\leq C\big(t^{-1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t(\|\boldsymbol{F}\|_1 + \|K\|_2)\big). \end{aligned} \tag{6.10}$$

We claim the above is enough for our purpose.

Proof of Theorem 6.1. Suppose momentarily that Theorem 6.2 is proved. Then estimate (6.3) can be obtained by combining (6.9) (for $\|P^E\|_1$), (6.10), and (6.8). Moreover,

$$\begin{aligned}
\|\boldsymbol{\Phi}^E\|_{3/2}^2 &\le C\|\boldsymbol{\Phi}^E\|_1\|\boldsymbol{\Phi}^E\|_2 \\
&\le C\big(t^{1/2}(\|\boldsymbol{F}\|_{-1/2}+\|K\|_{1/2})+t^{3/2}(\|\boldsymbol{F}\|_{1/2}+\|K\|_{3/2})\big) \\
&\qquad \cdot \big(t^{-1/2}(\|\boldsymbol{F}\|_{-1/2}+\|K\|_{1/2})+t(\|\boldsymbol{F}\|_1+\|K\|_2)\big) \\
&\le C(\|\boldsymbol{F}\|_{-1/2}^2+\|K\|_{1/2}^2+t^2\|\boldsymbol{F}\|_{1/2}^2+t^2\|K\|_{3/2}^2+t^3\|\boldsymbol{F}\|_1^2+t^3\|K\|_2) \\
&\le C\big(\|\boldsymbol{F}\|_{-1/2}^2+\|K\|_{1/2}^2+t^3\|\boldsymbol{F}\|_1^2+t^3\|K\|_2^2\big),
\end{aligned}$$

where we use the fact that

$$\|\boldsymbol{F}\|_{1/2} \le C(t^{1/2}\|\boldsymbol{F}\|_1+t^{-1}\|\boldsymbol{F}\|_{-1/2}), \qquad \|K\|_{3/2} \le C(t^{1/2}\|K\|_2+t^{-1}\|K\|_{1/2}).$$

So

$$\|\boldsymbol{\Phi}^E\|_{3/2} \le C\big(\|\boldsymbol{F}\|_{-1/2}+\|K\|_{1/2}+t^{3/2}(\|\boldsymbol{F}\|_1+\|K\|_2)\big).$$

Similarly

$$\|P^E\|_{1/2} \le C\big(\|\boldsymbol{F}\|_{-1/2}+\|K\|_{1/2}+t(\|\boldsymbol{F}\|_{1/2}+\|K\|_{3/2})\big),$$

and

$$\|P^E\|_{3/2} \le C\big(t^{-1}(\|\boldsymbol{F}\|_{-1/2}+\|K\|_{1/2})+t^{1/2}(\|\boldsymbol{F}\|_1+\|K\|_2)\big).$$

Combining these estimates on $\boldsymbol{\Phi}^E$ and $\mathcal{P}^E$ with the estimates in (6.8) for $\boldsymbol{\Phi}_0$ and P_0 then gives (6.4) and (6.5). □

Therefore it remains to prove Theorem 6.2. From the definitions we get

$$(\mathcal{C}\,\mathcal{E}(\boldsymbol{\Phi}^E),\mathcal{E}(\boldsymbol{\Psi}))-(\boldsymbol{\Psi},\mathbf{curl}\,P^E)=-\langle M_{\boldsymbol{s}}\boldsymbol{\Phi}_0,\boldsymbol{\Psi}\cdot\boldsymbol{s}\rangle \quad \text{for all } \boldsymbol{\Psi}\in\boldsymbol{H}^1, \tag{6.11}$$

$$-(\boldsymbol{\Phi}^E+\lambda^{-1}t^2\,\mathbf{curl}\,P^E,\mathbf{curl}\,Q)=\lambda^{-1}t^2(\mathbf{curl}\,P_0,\mathbf{curl}\,Q) \quad \text{for all } Q\in H^1. \tag{6.12}$$

We will prove Theorem 6.2 by choosing the appropriate test functions in these equations. First we need a lemma.

Lemma 6.3. *Under the same conditions of Theorem 6.1, there is a constant C such that for $r \in H^1(\partial\Omega)$*

$$\begin{aligned}|\langle \boldsymbol{\Phi}^E \cdot \boldsymbol{s}, r\rangle| \leq & Ct^{3/2}(\|r\|_{0,\partial\Omega} + t\|r\|_{1,\partial\Omega})(\|\boldsymbol{F}\|_0 + \|K\|_1 + \|P^E\|_1) \\ & + Ct^{1/2}\|r\|_{0,\partial\Omega}\|\boldsymbol{\Phi}^E\|_1.\end{aligned}$$

Proof of Lemma 6.3. We define the usual boundary-fitted coordinates in a neighborhood of the boundary. Let ρ_0 be a positive number less than the minimum radius of curvature of $\partial\Omega$ and define

$$\Omega_0 = \{\, \boldsymbol{z} - \rho \boldsymbol{n}_z | \boldsymbol{z} \in \partial\Omega,\ 0 < \rho < \rho_0 \,\},$$

where $\boldsymbol{n}_z$ is the outward unit normal to Ω at $\boldsymbol{z}$. Let $\boldsymbol{z}(\theta) = (X(\theta), Y(\theta))$, $\theta \in [0, L)$, be a parametrization of $\partial\Omega$ by arclength which we extend $L-$periodically to $\theta \in \mathbb{R}$. The correspondence

$$(\rho, \theta) \to \boldsymbol{z} - \rho \boldsymbol{n}_z = (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta))$$

is a diffeomorphism of $(0, \rho_0) \times \mathbb{R}/L$ on Ω_0. For any function f , let $\hat{f}(\rho, \theta)$ denote the change of variable to the (ρ, θ)-coordinate.

Now, we define an extension R of r to Ω_0 by

$$R(\rho, \theta) = \hat{r}(\theta) e^{-\rho/t}.$$

Then find a smooth cut-off function χ which is a function of ρ alone, independent of θ and t, and identically one for $0 \leq \rho \leq \rho_0/3$, identically zero for $\rho > 2\rho_0/3$. Thus χR gives an extension to all of Ω and, by simple computations,

$$\|\chi R\|_0 \leq Ct^{1/2}\|r\|_{0,\partial\Omega}, \quad \|\chi R\|_1 \leq C(t^{-1/2}\|r\|_{0,\partial\Omega} + t^{1/2}\|r\|_{1,\partial\Omega}).$$

Using integration by parts and (6.12) with $Q = \chi R$ we obtain

$$\begin{aligned}\langle \boldsymbol{\Phi}^E \cdot \boldsymbol{s}, r\rangle &= (\mathbf{curl}(\chi R), \boldsymbol{\Phi}^E) - (\chi R, \operatorname{rot} \boldsymbol{\Phi}^E) \\ &= -\lambda^{-1} t^2 (\mathbf{curl}(P^E + P_0), \mathbf{curl}(\chi R)) - (\chi R, \operatorname{rot} \boldsymbol{\Phi}^E)\end{aligned}$$

Applying the Schwartz inequality and the estimates on χR leads to the proof of the lemma. □

We are now in the position to prove Theorem 6.2.

Proof of Theorem 6.2. Taking $\boldsymbol{\Psi} = \boldsymbol{\Phi}^E$ in (6.11) and $Q = P^E$ in (6.12) gives

$$\begin{aligned}(C\,\mathcal{E}(\boldsymbol{\Phi}^E), \mathcal{E}(\boldsymbol{\Phi}^E)) &+ \lambda^{-1} t^2 (\mathbf{curl}\, P^E, \mathbf{curl}\, P^E) \\ &= -\langle M_{\boldsymbol{s}} \boldsymbol{\Phi}_0, \boldsymbol{\Phi}^E \cdot \boldsymbol{s}\rangle - \lambda^{-1} t^2 (\mathbf{curl}\, P_0, \mathbf{curl}\, P^E).\end{aligned} \tag{6.13}$$

We bound the first term on the right hand side using Lemma 6.3 and the bounds (6.8) on $\boldsymbol{\Phi}_0$:

$$\begin{aligned}&|\langle M_{\boldsymbol{s}} \boldsymbol{\Phi}_0, \boldsymbol{\Phi}^E \cdot \boldsymbol{s}\rangle| \\ &\le C(t^{1/2} \|M_{\boldsymbol{s}} \boldsymbol{\Phi}_0\|_{0,\partial\Omega} + t^{3/2} \|M_{\boldsymbol{s}} \boldsymbol{\Phi}_0\|_{1,\partial\Omega})(t\|\boldsymbol{F}\|_0 + t\|K\|_1 + t\|P^E\|_1) \\ &\quad + C t^{1/2} \|M_{\boldsymbol{s}} \boldsymbol{\Phi}_0\|_{0,\partial\Omega} \|\boldsymbol{\Phi}^E\|_1 \\ &\le C\big(\, t^{1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t^{3/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\,\big) \\ &\quad \cdot \big(\, t\|\boldsymbol{F}\|_0 + t\|K\|_1 + t\|P^E\|_1\,\big) + C t^{1/2} (\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) \|\boldsymbol{\Phi}^E\|_1.\end{aligned} \tag{6.14}$$

For the second term on the right hand side of (6.13) we have

$$|t^2 (\mathbf{curl}\, P_0, \mathbf{curl}\, P^E)| \le C t (\|\boldsymbol{F}\|_0 + \|K\|_1)\, t \|P^E\|_1. \tag{6.15}$$

Next we choose Q with zero average and $\mathbf{curl}\, Q = \boldsymbol{P}\boldsymbol{\Phi}^E$, the L^2-projection onto the rigid motions (the space spanned by $\{\, (a - by, c + bx) | a, b, c, \in \mathbb{R}\,\}$) in (6.12), to get

$$(\boldsymbol{\Phi}^E, \boldsymbol{P}\boldsymbol{\Phi}^E) = -\lambda^{-1} t^2 (\mathbf{curl}(P^E + P_0), \boldsymbol{P}\boldsymbol{\Phi}^E),$$

which implies

$$\|\boldsymbol{P}\boldsymbol{\Phi}^E\|_0 \le Ct(t\|\,\mathbf{curl}\,P^E\|_0 + t\|\boldsymbol{F}\|_0 + t\|K\|_1). \tag{6.16}$$

Combining (6.13)–(6.16) and using the equivalence between $\|\boldsymbol{\phi}\|_1$ and $|(C\mathcal{E}(\boldsymbol{\phi}),\mathcal{E}(\boldsymbol{\phi}))|^{1/2} + \|\boldsymbol{P}\boldsymbol{\phi}\|_0$ gives

$$\|\boldsymbol{\Phi}^E\|_1 + t\|P^E\|_1 \le C\big(t^{1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t^{3/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\big),$$

where we use the fact that

$$\|\boldsymbol{F}\|_0 \le C(t^{1/2}\|\boldsymbol{F}\|_{1/2} + t^{-1/2}\|\boldsymbol{F}\|_{-1/2}), \qquad \|K\|_1 \le C(t^{1/2}\|K\|_{3/2} + t^{-1/2}\|K\|_{1/2}).$$

Finally we choose $\boldsymbol{\Psi}$ in (6.11) with

$$\operatorname{rot}\boldsymbol{\Psi} = P^E, \qquad \boldsymbol{\Psi}\cdot\boldsymbol{s} = 0 \quad \text{on } \partial\Omega, \qquad \|\boldsymbol{\Psi}\|_1 \le C\|P^E\|_0$$

to get

$$(P^E, P^E) = (C\mathcal{E}(\boldsymbol{\Phi}^E), \mathcal{E}(\boldsymbol{\Psi})).$$

Thus

$$\|P^E\|_0 \le C\|\boldsymbol{\Phi}^E\|_1.$$

This completes the proof of the first estimate of Theorem 6.2.

To get the second estimate we use elliptic regularity. From (6.11) we see

$$-\,\mathbf{div}\,C\,\mathcal{E}(\boldsymbol{\Phi}^E) = \mathbf{curl}\,P^E \quad \text{in } \Omega,$$

$$M_{\boldsymbol{s}}\boldsymbol{\Phi}^E = -M_{\boldsymbol{s}}\boldsymbol{\Phi}_0, \quad M_{\boldsymbol{n}}\boldsymbol{\Phi}^E = 0 \quad \text{on } \partial\Omega.$$

Therefore

$$\begin{aligned}\|\boldsymbol{\Phi}^E\|_2 &\le C\big(\,\|P^E\|_1 + \|M_{\boldsymbol{s}}\boldsymbol{\Phi}_0\|_{1/2,\partial\Omega} + \|\boldsymbol{P}\boldsymbol{\Phi}^E\|_0\,\big)\\ &\le C\big(\,t^{-1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t^{1/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2}) + \|\boldsymbol{F}\|_0 + \|K\|_1\,\big)\\ &\le C\big(\,(t^{-1/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) + t^{1/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\big),\end{aligned}$$

as desired.

Similarly, by (6.12)

$$
\begin{aligned}
-\Delta P^E &= \Delta P_0 - \lambda t^{-2} \operatorname{rot} \mathbf{\Phi}^E \quad && \text{in } \Omega, \\
\frac{\partial P^E}{\partial n} &= -\frac{\partial P_0}{\partial n} - \lambda t^{-2} \mathbf{\Phi}^E \cdot \boldsymbol{s} && \text{on } \partial\Omega,
\end{aligned}
$$

so

$$
\begin{aligned}
\|P^E\|_2 &\le C(\|P_0\|_2 + t^{-2}\|\mathbf{\Phi}^E\|_1) \\
&\le C\big(\|\boldsymbol{F}\|_1 + \|K\|_2 + t^{-3/2}(\|\boldsymbol{F}\|_{-1/2} + \|K\|_{1/2}) \\
&\quad + t^{-1/2}(\|\boldsymbol{F}\|_{1/2} + \|K\|_{3/2})\big),
\end{aligned}
$$

which is the desired estimate on $\|P^E\|_2$. $\square$

Combining the standard interpolation theory and Theorem 6.1, we get

Corollary 6.4. *Under the same conditions of Theorem 6.1, we have*

$$\|\mathbf{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon}\big(\|\boldsymbol{F}\|_0 + t^{3/2}\|\boldsymbol{F}\|_1\big) \quad \textit{for} \quad K = 0, \tag{6.17}$$

$$\|\mathbf{\Phi}\|_{3/2+\epsilon} + \|P\|_{1/2+\epsilon} + t\|P\|_{3/2+\epsilon} \le C_\epsilon t^{-\epsilon}\|K\|_2 \quad \textit{for} \quad \boldsymbol{F} = 0, \tag{6.18}$$

for $0 \le \epsilon \le 1/2$.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 21 (1984), pp. 337–344.

[3] D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Num. Anal., 26 (1989), pp. 1276–1290.

[4] ———, *The boundary layer for Reissner-Mindlin plate*, SIAM J. Math. Anal, 26 (1990), pp. 10–20.

[5] ———, *The boundary layer for the Reissner-Mindlin plate model: soft simply supported, soft clamped and free plates*, to appear.

[6] D. N. Arnold and X. Liu, *Local error estimates for finite element discretizations of the Stokes equations*, to appear.

[7] D. N. Arnold, private communication.

[8] D. N. Arnold and R. S. Falk, *Edge Effects in the Reissner-Mindlin Plate Theory*, Analytic and Computation Models of Shells, 1989.

[9] I. Babuška and R. Rodríguez, *The problem of the selection of an a-posteriori error indicator based on smoothing techniques*, Institute for Physical Science and Technology, BN-1126 (August 1991).

[10] R. E. Bank and A. Weisser, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp., 44 (1985), pp. 283–301.

[11] K. J. Bathe, *Finite element procedure in engineering analysis*, Prentice-Hall, 1982.

[12] J. H. Bramble and A. H. Schatz, *Higher order local accuracy by local averaging in the finite element method*, Math. Comp., 31 (1977), pp. 94–111.

[13] F. Brezzi, *On the existence, uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers* , RAIRO Anal. Numer., 8 (1974), pp. 129–151.

[14] F. Brezzi, K. J. Bathe, and A. Fortin, *Mixed-Interpolated elements for Reissner-Mindlin plates*, Int. J. Num. Meth. in Eng., 28 (1989), pp. 1787–1802.

[15] F. Brezzi and A. Fortin, *Numerical approximation of Mindlin-Reissner plates*, Math. Comp., 47 (1986), pp. 151–158.

[16] P. G. Ciarlet, *The Finite Element Method for Elliptic Equations*, North-Holland, Amsterdam , 1978.

[17] ———, *Finite element methods for elliptic boundary value problems*, in Handbook of Numerical Analysis Volume II, P. G. Ciarlet and J. L. Lions, eds., Elsevier, Amsterdam–New York, 1991.

[18] M. Crouzeix and P-A. Raviart, *Conforming and non-conforming finite element methods for solving the stationary Stokes equations*, RAIRO Anal. Numér., 7 R-3 (1973), pp. 33–76.

[19] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- or three-dimentional domains with corners. Part I: Linearized equations*, SIAM J. Math. Anal, 20 (1989), pp. 74–97.

[20] J. Douglas, Jr. and F. A. Milner, *Interior and superconvergence estimates for mixed methods for second order elliptic problems*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 397–428.

[21] J. Douglas, Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic problems*, Math. Comp., 44 (1985), pp. 39–52.

[22] R. Durán, M. A-M. Muschietti, and R. Rodríguez, *On the asymptotic exactness of error estimators for linear traingular finite elements*, Numer. Math., 59 (1991), pp. 107–127.

[23] R. Durán and R. Rodríguez, *On the asymptotic exactness of Bank-Weisser's estimator*, Numer. Math., 62 (1992), pp. 292–303.

[24] E. Eriksson, *Higher order local rate of convergence by mesh refinement in the finite element method*, Math. Comp., 44 (1985), pp. 321–344.

[25] ———, *Improved accurccy by adapted mesh refinement in the finte element method*, Math. Comp., 45 (1985), pp. 109–142.

[26] E. Eriksson and C. Johnson, *An Adaptive Finite Element Method for Linear Elliptic Problems*, Math. Comp., 50 (1988), pp. 361–383.

[27] ———, *Adaptive finite element methods for parabolic problems I: A linear model problem*, SIAM J. Num. Anal, 28 (1991), pp. 43–77.

[28] L. Gastaldi, *Uniform interior error estimates for reissner-mindlin plate model*, preprint.

[29] P. Grisvard, *Problèmes aux limits dans les polygones. Mode d'emploi*, 1, E.D.F. Bulletin de la direction des Etudes et Recherches Série C. Mathématiques, Informatiques, 1986.

[30] T. R. Hughes, *The finite element method, Linear Static and Dynamic analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

[31] C. Johnson, in Numerical solutions of partial differential equations by the finite element method, Cambridge University Press, New York-New Rochelle-Melbourne-Sydney, 1987.

[32] J. L. Lions and E. Magnes, *Nonhomogeneous boundary value problems and applications*, Springer-Verlag, New York–Heidelberg–Berlin, 1972.

[33] J. A. Nitsche and A. H. Schatz, *Interior estimate for Ritz-Galerkin methods*, Math. Comp., 28 (1974), pp. 937–958.

[34] A. H. Schatz, *Lecture Notes in Mathematics 983*, Springer-Verlag, New York–Heidelberg–Berlin, 1986.

[35] A. H. Schatz and L. B. Wahlbin, *Interior Maximun norm estimates for finite telement methods*, MATHC3, 31 (1977), pp. 414–442.

[36] ———, *Maximun norm estimates in the finite telement methods on plane polygonal domain. Part 1*, Math. Comp., 32 (1978), pp. 73–109.

[37] ———, *Maximun norm estimates in the finite telement methods on plane polygonal domain. Part 2 Refinements*, Math. Comp., 33 (1979), pp. 465–492.

[38] ———, *On the finite element for singularly perturbed reaction-diffusion problems in two and one dimensions*, Math. Comp., 40 (1983), pp. 47–89.

[39] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying bounday conditions*, Math. Comp., 54 (1990), pp. 483–493.

[40] R. Temam, *Navier-Stokes Equations*, Amsterdam, Amsterdam, 1984.

[41] L. B. Wahlbin, *Local Behavior in Finite Element Methods*, in in Handbook of Numerical Analysis Volume II, P. G. Ciarlet and J. L. Lions, eds., Elsevier, Amsterdam–New York, 1991.

[42] M. F. Wheeler and J. R. Whiteman, *Superconvergence recovery of gradients on subdomains from piecewise linear finite-elemen approximations*, Numerical methods for partial differential equations, 3 (1987), pp. 357–374.

[43] J. Xu, *Theorey of Multilevel methods*, Tech. Report # 48, Department of Mathematics, the Penn State University, 1989.

[44] C. Zhan and L. Wang, *The interior Estimates for Nonconforming Finite elements*, 12.4, Chiness J. Num. Math. & Appl., 1990.

[45] J. Z. Zhu and O. C. Zienkiewicz, *Adaptive techniques in the finite element method*, Comm. in Appl. Num. methods, 4 (1988), pp. 197–204.

[46] O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, Internat. J. Numer. Methods Engrg., 24 (1987), pp. 337–357.