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Multigrid in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ ⋆

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Dedicated to Olof B. Widlund on the occasion of his 60th birthday.

Summary. We consider the solution of systems of linear algebraic equations which arise from the finite element discretization of variational problems posed in the Hilbert spaces $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ in three dimensions. We show that if appropriate finite element spaces and appropriate additive or multiplicative Schwarz smoothers are used, then the multigrid V-cycle is an efficient solver and preconditioner for the discrete operator. All results are uniform with respect to the mesh size, the number of mesh levels, and weights on the two terms in the inner products.

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1. Introduction

The Hilbert spaces $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ consist of square-integrable vector fields on a domain $\Omega \subset \mathbb{R}^3$ with square-integrable divergence and curl, respectively. Define bilinear forms

$$\Lambda^{\mathrm{d}}(\boldsymbol{u},\boldsymbol{v}) = \rho^2(\boldsymbol{u},\boldsymbol{v}) + \kappa^2(\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v}),$$
$$\Lambda^{\mathrm{c}}(\boldsymbol{p},\boldsymbol{q}) = \rho^2(\boldsymbol{p},\boldsymbol{q}) + \kappa^2(\mathbf{curl}\,\boldsymbol{p}, \mathbf{curl}\,\boldsymbol{q}),$$

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where on the right hand sides $(\cdot, \cdot)$ denotes the inner product in $\boldsymbol{L}^2 = \boldsymbol{L}^2(\Omega)$ and ρ and κ are positive parameters. For $\rho = \kappa = 1$, these are precisely the inner products in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, respectively, and for any positive ρ and κ, they define equivalent inner products. We stress from the outset that the results in this paper hold uniformly for $0 < \rho, \kappa < \infty$.

Given a finite element subspace $\boldsymbol{V}_h$ of $\boldsymbol{H}(\mathrm{div})$, we determine a positive definite symmetric operator $\boldsymbol{\Lambda}_h^{\mathrm{d}} : \boldsymbol{V}_h \to \boldsymbol{V}_h$ by $(\boldsymbol{\Lambda}_h^{\mathrm{d}}\boldsymbol{u}, \boldsymbol{v}) = \Lambda^{\mathrm{d}}(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h$, and, analogously, given a finite element subspace $\boldsymbol{Q}_h$ of $\boldsymbol{H}(\mathbf{curl})$, we determine a positive definite symmetric operator $\boldsymbol{\Lambda}_h^{\mathrm{c}} : \boldsymbol{Q}_h \to \boldsymbol{Q}_h$ by $(\boldsymbol{\Lambda}_h^{\mathrm{c}}\boldsymbol{p}, \boldsymbol{q}) = \Lambda^{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{q})$ for all $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{Q}_h$. These operators are discretizations of $\boldsymbol{\Lambda}^{\mathrm{d}} := \rho^2\boldsymbol{I} - \kappa^2\mathbf{grad}\,\mathrm{div}$ and $\boldsymbol{\Lambda}^{\mathrm{c}} := \rho^2\boldsymbol{I} + \kappa^2\mathbf{curl}\,\mathbf{curl}$, respectively, with natural boundary conditions. For any $\boldsymbol{f} \in \boldsymbol{V}_h$ and $\boldsymbol{g} \in \boldsymbol{Q}_h$, the equations

$$(1.1) \qquad \boldsymbol{\Lambda}_h^{\mathrm{d}}\boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{\Lambda}_h^{\mathrm{c}}\boldsymbol{p} = \boldsymbol{g},$$

admit unique solutions $\boldsymbol{u} \in \boldsymbol{V}_h$ and $\boldsymbol{p} \in \boldsymbol{Q}_h$, respectively. In this paper we study the efficient solution of these equations using multigrid.

The spaces $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ arise naturally in many problems of fluid mechanics, solid mechanics, and electromagnetism. Frequently these applications require a fast solution method for one or both of the equations 1.1. In some applications, essential boundary conditions are imposed. That is, the bilinear form Λ^{d} is restricted to $\mathring{\boldsymbol{H}}(\mathrm{div})$, the subspace of $\boldsymbol{H}(\mathrm{div})$ consisting of vector fields whose normal component vanishes on $\partial\Omega$, or the bilinear form Λ^{c} is restricted to $\mathring{\boldsymbol{H}}(\mathbf{curl})$, the subspace of $\boldsymbol{H}(\mathbf{curl})$ consisting of vector fields whose tangential component vanishes on $\partial\Omega$. Although we will not treat this situation explicitly here, the results and analysis we give adapt to the case of essential boundary conditions with only minor and straightforward modifications.

In Sect.7 of [2], we discuss in detail the application of fast solvers for the equation $\boldsymbol{\Lambda}_h^{\mathrm{d}}\boldsymbol{u} = \boldsymbol{f}$ to both mixed and least squares formulations of second order elliptic boundary value problems, including one in which $\kappa \ll 1$. Several other applications are discussed briefly as well. Applications of fast solvers for $\boldsymbol{\Lambda}_h^{\mathrm{c}}\boldsymbol{p} = \boldsymbol{g}$ arise in various contexts in electromagnetism. For example, in simple time-discretizations of Maxwell's equations, this occurs with κ proportional to the time step. See [12] for a detailed discussion. Such solvers also have applications to some formulations of the Navier–Stokes equations as discussed in [8] and [9].

Multigrid methods have been established as among the most efficient solvers for discretized elliptic problems and a considerable theory has been developed to justify their use. See, e.g., [5,10,18]. Unfortunately, some of the simplest and most frequently used smoothers for elliptic problems do not yield effective multigrid iterations when applied to the problems considered here (see, for example, [6]). This failure can be traced to a key difference between the operators $\boldsymbol{\Lambda}^{\mathrm{d}}$ and $\boldsymbol{\Lambda}^{\mathrm{c}}$ on the one hand and elliptic operators on the other. Namely, the eigenspace associated to the least eigenvalue of the former operators contains many eigenfunctions which cannot be represented well on a coarse mesh (while low eigenvalue eigenfunctions for standard elliptic operators are always slowly varying). This is because the operator $\boldsymbol{\Lambda}^{\mathrm{d}}$ reduces to the identity when applied to solenoidal vector fields, although it behaves like a second order elliptic operator when applied to irrotational vector fields. Exactly the reverse holds for $\boldsymbol{\Lambda}^{\mathrm{c}}$. It is therefore not surprising that the Helmholtz decomposition of an arbitrary vector field into irrotational and solenoidal components plays an important role in the understanding and analysis of these problems. In particular, we make substantial use of discrete versions of the Helmholtz decomposition in our analysis of multigrid methods.

The main result of this paper is a proof that the standard V-cycle multigrid algorithm is an effective solver or preconditioner for problems involving the operators $\boldsymbol{\Lambda}_h^{\mathrm{d}}$ or $\boldsymbol{\Lambda}_h^{\mathrm{c}}$ in three dimensions if (1) appropriate finite element subspaces of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ are taken, and (2) appropriate smoothers are used. More precisely, we show that if we take $\boldsymbol{V}_h$ to be the Raviart–Thomas–Nedelec space of any order with respect to a tetrahedral mesh of size h, and if $\boldsymbol{\Theta}_h^{\mathrm{d}}$ is the approximate inverse of $\boldsymbol{\Lambda}_h^{\mathrm{d}}$ defined by the V-cycle algorithm using any of several additive or multiplicative Schwarz smoothers, then $\boldsymbol{I} - \boldsymbol{\Theta}_h^{\mathrm{d}}\boldsymbol{\Lambda}_h^{\mathrm{d}}$ is a positive definite contraction, whose norm is bounded away from 1 uniformly in the mesh size h, the number of mesh levels, and the parameters $\rho, \kappa \in (0, \infty)$. Of course, this implies that $\boldsymbol{\Theta}_h^{\mathrm{d}}$ is a good preconditioner as well: the condition number of $\boldsymbol{\Theta}_h^{\mathrm{d}}\boldsymbol{\Lambda}_h^{\mathrm{d}}$ is bounded independently of h, the number of mesh levels, and ρ and κ. To define the Schwarz smoothers, we can use a decomposition of $\boldsymbol{V}_h$ into local patches consisting of all elements surrounding either an edge or a vertex, or a third decomposition can be used based on the Helmholtz decomposition (see 4.2). Precisely analogous results hold in the $\boldsymbol{H}(\mathbf{curl})$ case if we take $\boldsymbol{Q}_h$ to be the Nedelec edge spaces of any order. In this case, the smoothers can be based either on a decomposition based on vertex patches or on the decomposition 4.4 arising from the Helmholtz decomposition.

The results of this paper generalize to three dimensions ones which we obtained for $\boldsymbol{H}(\mathrm{div})$ in two dimensions [2]. The spaces $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ are essentially the same in two dimensions, and so our analysis of multigrid in [2] adapts to $\boldsymbol{H}(\mathbf{curl})$ with only the most mechanical changes. In three dimensions, while there are many similarities between $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, there are also significant differences, especially between their finite element discretizations. For this reason, the analysis for $\boldsymbol{H}(\mathbf{curl})$ requires a number of additional ideas. In our presentation, we have stressed the similarity between the $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ cases as much as possible, limiting the differences to the proofs of the two-level estimates for mixed methods in the final section, where they are described explicitly.

The first results for multigrid in $\boldsymbol{H}(\mathrm{div})$ in three dimensions are due to Hiptmair in [11]. The same author obtained the first results for multigrid in $\boldsymbol{H}(\mathbf{curl})$ in [12]. A unified and simplified treatment of those important works is given by Hiptmair and Toselli in [13]. Our results are closely related to the results in [11–13], and some of our arguments derive from them. The major difference between our approach and theirs is that we employ a multigrid framework as presented, for example, in [5], and verify the hypotheses required by this approach by developing necessary estimates for mixed finite element methods based on discretizations of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. Specifically, we take Theorem 3.1 below as the basis for our analysis, and develop *two-level estimates* for mixed methods in Sect.5 in order to apply this theorem. By contrast, Hiptmair and Toselli use an overlapping Schwarz method framework as presented, for example in [16]. An important benefit of our approach, which is also somewhat less complicated, is that we obtain estimates which are independent of the parameters ρ and κ occuring in the bilinear form. By contrast, in [12], the condition number of $\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h$ is only shown to be $O(1/\kappa^3)$ when $\rho = 1$ and κ is small (and the case of κ/ρ large is not discussed).

Concerning notation, we use boldface type for vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. The norm in the Sobolev spaces $H^s(\Omega)$ and $\boldsymbol{H}^s(\Omega)$ are both denoted by $\|\cdot\|_s$, with the index $s = 0$ suppressed. The norm associated to the bilinear form Λ^{d} is denoted $\|\cdot\|_{\Lambda^{\mathrm{d}}}$, or simply $\|\cdot\|_{\boldsymbol{H}(\mathrm{div})}$ if $\rho = \kappa = 1$, and analogously for the norm associated to Λ^{c}.

We conclude the section with an outline of the remainder of the paper. In the next section, we introduce the finite dimensional subspaces of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ that we shall consider in this paper, the Raviart–Thomas–Nedelec spaces and Nedelec edge spaces, respectively. We then state some of the key properties of these spaces which we shall use in the subsequent analysis, the most important of which are discrete Helmholtz decompositions of each space. In Sect.3, we state some standard results for multigrid iterations, in order to isolate sufficient conditions on additive and multiplicative Schwarz smoothers for efficient algorithms. In Sect.4, we apply these result to obtain the convergence of the standard multigrid V-cycle for the operators $\boldsymbol{\Lambda}_h^{\mathrm{d}}$ and $\boldsymbol{\Lambda}_h^{\mathrm{c}}$ using appropriately defined additive and multiplicative Schwarz smoothers. The proof hinges on certain two-level error estimates for mixed methods based on the Raviart–Thomas–Nedelec and Nedelec edge spaces. These estimates are stated and proved in Sect. 5.

2. Finite element discretization

We suppose that Ω is a bounded and convex polyhedron in $\mathbb{R}^3$ and $\mathcal{T}_h$ a mesh of Ω consisting of closed tetrahedra. We assume that the mesh is shape regular and quasi-uniform. More precisely, the constants that appear in the estimates below may depend on the shape regularity constant (the maximum ratio of the diameter of an element to the diameter of the largest ball contained in the element) and the quasi-uniformity constant (the maximum ratio of the largest to the smallest element diameter) of the mesh, but are otherwise mesh-independent. We denote by $\mathcal{V}_h$, $\mathcal{E}_h$, and $\mathcal{F}_h$ the sets of vertices, edges, and faces of the mesh, respectively. For $\nu \in \mathcal{V}_h \cup \mathcal{E}_h \cup \mathcal{F}_h \cup \mathcal{T}_h$ we define

$$\mathcal{T}_h^\nu = \{\, T \in \mathcal{T}_h \,|\, \nu \subset T \,\}, \quad \Omega_h^\nu = \text{interior}(\bigcup \mathcal{T}_h^\nu).$$

Thus Ω_h^ν is the subdomain of Ω formed by the patch of elements meeting at ν, and $\mathcal{T}_h^\nu$ is the restriction of the mesh $\mathcal{T}_h$ to Ω_h^ν.

Fix an integer $k \geq 0$. We then recall the following spaces (the spaces $\boldsymbol{Q}_h$ and $\boldsymbol{V}_h$ were introduced in [15]; see also [3] and [14] for the connection between these spaces and differential forms, which illuminates many of their properties):

W_h : continuous piecewise polynomials of degree at most $k+1$,
$\boldsymbol{Q}_h$: the Nedelec edge discretization of $\boldsymbol{H}(\mathbf{curl})$ of index k,
$\boldsymbol{V}_h$: the Raviart–Thomas–Nedelec discretization of $\boldsymbol{H}(\mathrm{div})$ of index k,
S_h : arbitrary piecewise polynomials of degree at most k.

To define these spaces, we specify the corresponding polynomial spaces used on each element and the corresponding sets of degrees of freedom. Restricted to a tetrahedron T, the elements of W_h and S_h are, of course, arbitrary elements of $\mathcal{P}_{k+1}(T)$ and $\mathcal{P}_k(T)$, respectively, where $\mathcal{P}_k(T)$ denotes the space of polynomials of degree at most k restricted to T. The restrictions of the elements of $\boldsymbol{V}_h$ are functions of the form $\boldsymbol{p}(\boldsymbol{x})+r(\boldsymbol{x})\boldsymbol{x}$ with $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_k(T)$ and $r \in \mathcal{P}_k(T)$. The elements of $\boldsymbol{Q}_h$ are functions of the form $\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x})$ with $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_k(T)$ and $\boldsymbol{r} \in \boldsymbol{\mathcal{P}}_{k+1}(T)$ such that $\boldsymbol{r} \cdot \boldsymbol{x} \equiv 0$. The degrees of freedom for $\boldsymbol{u} \in \boldsymbol{V}_h$ are of two sorts. First, the moments of $\boldsymbol{u} \cdot \boldsymbol{n}$ of order at most k on each face f (more precisely the functionals that associate to $\boldsymbol{u}$ its inner product in $L^2(f)$ with each element of a basis for $\mathcal{P}_k(f)$); and second, the moments of $\boldsymbol{u}$ of degree $k-1$ on each tetrahedron. The degrees of freedom of $\boldsymbol{q} \in \boldsymbol{Q}_h$ are (1) the moments of $\boldsymbol{q} \cdot \boldsymbol{s}$ of order at most k on each edge, (2) the moments of $\boldsymbol{q} \times \boldsymbol{n}$ of order at most $k-1$ on each face, and (3) the moments of $\boldsymbol{q}$ of order at most $k-2$ on each tetrahedron. For S_h we use as degrees of freedom the tetrahedral moments of order at most k. For W_h, we use (1) the values at the vertices, (2) the edge moments of

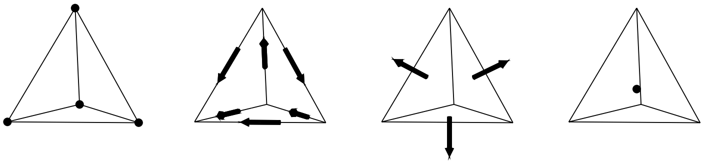

Fig. 1. Degrees of freedom for the spaces W_h, $\boldsymbol{Q}_h$, $\boldsymbol{V}_h$, and S_h in the lowest order case $k = 0$.

order at most $k-1$, (3) the face moments of order at most $k-2$, and (4) the tetrahedral moments of order at most $k-3$.

We now consider the decomposition of these spaces as sums of spaces supported in small patches of elements. Define

$$\boldsymbol{Q}_h^\nu = \{\boldsymbol{r} \in \boldsymbol{Q}_h \,:\, \operatorname{supp}\boldsymbol{r} \subset \bar{\Omega}_h^\nu\}, \quad \nu \in \mathcal{V}_h \cup \mathcal{E}_h \cup \mathcal{F}_h \cup \mathcal{T}_h,$$

and analogously with $\boldsymbol{Q}_h$ replaced by W_h, $\boldsymbol{V}_h$, or S_h. Then

$$\begin{aligned} W_h &= \textstyle\sum_{v\in\mathcal{V}_h} W_h^v, \\ \boldsymbol{Q}_h &= \textstyle\sum_{v\in\mathcal{V}_h} \boldsymbol{Q}_h^v = \sum_{e\in\mathcal{E}_h} \boldsymbol{Q}_h^e, \\ \boldsymbol{V}_h &= \textstyle\sum_{v\in\mathcal{V}_h} \boldsymbol{V}_h^v = \sum_{e\in\mathcal{E}_h} \boldsymbol{V}_h^e = \sum_{f\in\mathcal{F}_h} \boldsymbol{V}_h^f, \\ S_h &= \textstyle\sum_{v\in\mathcal{V}_h} S_h^v = \sum_{e\in\mathcal{E}_h} S_h^e = \sum_{f\in\mathcal{F}_h} S_h^f = \sum_{T\in\mathcal{T}_h} S_h^T. \end{aligned} \tag{2.1}$$

For each of these decompositions there is a corresponding estimate on the sum of the squares of the L^2 norms of the summands. For example, we can decompose an arbitrary element $\boldsymbol{q} \in \boldsymbol{Q}_h$ as $\boldsymbol{q} = \sum_{e\in\mathcal{E}_h} \boldsymbol{q}^e$ with $\boldsymbol{q}^e \in \boldsymbol{Q}_h^e$ so that the estimate

$$\sum_{e\in\mathcal{E}_h} \|\boldsymbol{q}^e\|^2 \le c\|\boldsymbol{q}\|^2 \tag{2.2}$$

holds with c depending only on the shape regularity of the mesh. We remark also that the decompositions not stated in fact don't hold. For example, $W_h \neq \sum_{e\in\mathcal{E}_h} W_h^e$.

The proof of these decompositions and the corresponding estimates all follow the same lines. For example, to prove the edge-based decomposition of $\boldsymbol{Q}_h$ and the estimate 2.2, we note that the degrees of freedom of the space $\boldsymbol{Q}_h$ determine a canonical decomposition of an arbitrary element $\boldsymbol{q} \in \boldsymbol{Q}_h$ as $\boldsymbol{q} = \sum \boldsymbol{q}^\xi$ where the sum runs over all the degrees of freedom of $\boldsymbol{Q}_h$, and $\boldsymbol{q}^\xi$ is the element of $\boldsymbol{Q}_h$ with all degrees of freedom other than ξ set equal to zero. A standard scaling argument then implies that $\|\boldsymbol{q}^\xi\| \le c\|\boldsymbol{q}\|_{L^2(\operatorname{supp}\boldsymbol{q}^\xi)}$. Now to each degree of freedom ξ, we may assign an edge e of the mesh such that $\operatorname{supp}\boldsymbol{q}^\xi \subset \Omega_h^e$ (for an edge-based degree of freedom, choose e to be that edge and for a face- or tetrahedron-based degree of freedom, choose e to be any edge contained in the face or tetrahedron). Combining the corresponding $\boldsymbol{q}^\xi$ gives the desired decomposition.

These degrees of freedom specified for the spaces W_h, $\boldsymbol{Q}_h$, $\boldsymbol{V}_h$, and S_h determine interpolation operators Π_h^W, $\boldsymbol{\Pi}_h^Q$, $\boldsymbol{\Pi}_h^V$, and Π_h^S mapping functions which are sufficiently smooth that the required function values and moments exist into the subspaces. Specifically, Π_h^W is a standard interpolation operator and is defined on continuous functions, and Π_h^S is the L^2-projection operator, defined on all L^2 functions. As the domain for $\boldsymbol{\Pi}_h^V$, we can choose $\boldsymbol{H}^1$. Moreover, a standard argument based on the Bramble–Hilbert lemma and scaling gives the error estimate

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h^V \boldsymbol{v}\| \le ch\|\boldsymbol{v}\|_1, \quad \boldsymbol{v} \in \boldsymbol{H}^1. \tag{2.3}$$

Because of the dependence on edge moments, the situation is more complicated for the operator $\boldsymbol{\Pi}_h^Q$. It is bounded on the space of $\boldsymbol{H}^1$ vector fields whose $\mathbf{curl}$ belongs to L^p, for any fixed $p \in (2, \infty]$. This follows from Lemma 4.7 of [1] and the Sobolev embedding theorem. In particular, it is defined for $\boldsymbol{H}^1$ vector fields whose $\mathbf{curl}$ belongs to $\boldsymbol{V}_h$. Moreover we have

$$\|\boldsymbol{q} - \boldsymbol{\Pi}_h^Q \boldsymbol{q}\| \le ch\|\boldsymbol{q}\|_1, \qquad \boldsymbol{q} \in \boldsymbol{H}^1 \text{ such that } \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{V}_h. \tag{2.4}$$

To show this, we follow [13]. First consider the case where the mesh consists of only the unit simplex $\hat{T}$. Let $\hat{\boldsymbol{Q}}$ and $\hat{\boldsymbol{V}}$ denote the corresponding spaces and $\hat{\boldsymbol{\Pi}}^Q$ the interpolant. Using the equivalence of norms in $\hat{\boldsymbol{V}}$, we get

$$\|\hat{\boldsymbol{\Pi}}^Q \hat{\boldsymbol{q}}\| \le c(\|\hat{\boldsymbol{q}}\|_1 + \|\mathbf{curl}\,\hat{\boldsymbol{q}}\|_{L^\infty}) \le c\|\hat{\boldsymbol{q}}\|_1$$

for all $\hat{\boldsymbol{q}} \in \boldsymbol{H}^1(\hat{T})$ such that $\mathbf{curl}\,\hat{\boldsymbol{q}} \in \hat{\boldsymbol{V}}$. A Bramble–Hilbert argument then gives $\|\hat{\boldsymbol{q}} - \hat{\boldsymbol{\Pi}}^Q \hat{\boldsymbol{q}}\| \le c|\hat{\boldsymbol{q}}|_1$ for $\hat{\boldsymbol{q}} \in \boldsymbol{H}^1(\hat{T})$ such that $\mathbf{curl}\,\hat{\boldsymbol{q}} \in \hat{\boldsymbol{V}}$ where now only the $\boldsymbol{H}^1$ seminorm appears on the right hand side. If we scale this estimate to a general simplex $T = F^{-1}\hat{T}$ with F affine, using the appropriate contravariant transform $\hat{\boldsymbol{q}} \mapsto (DF)^*(\hat{\boldsymbol{q}} \circ F)$, and add up over all the simplices in the mesh, we get 2.4.

The interpolation operators also satisfy the commutativity properties

$$\mathbf{curl}\,\boldsymbol{\Pi}_h^Q = \boldsymbol{\Pi}_h^V \mathbf{curl}\,, \quad \mathrm{div}\boldsymbol{\Pi}_h^V = \Pi_h^S \mathrm{div}, \quad \mathbf{grad}\,\Pi_h^W = \boldsymbol{\Pi}_h^Q \mathbf{grad}$$

when applied to sufficiently smooth vector fields. These well-known relations follow from the definitions of the interpolation operators and the theorems of Green and Stokes.

In addition to these interpolation operators, we also define $\boldsymbol{P}_h^{\mathrm{d}}$ to be the orthogonal projection onto $\boldsymbol{V}_h$ with respect to the inner product in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{P}_h^{\mathrm{c}}$ to be the orthogonal projection onto $\boldsymbol{Q}_h$ with respect to the inner product in $\boldsymbol{H}(\mathbf{curl})$.

A key property relating the spaces W_h, $\boldsymbol{Q}_h$, $\boldsymbol{V}_h$, and S_h, is that (for a convex, or more generally, contractible domain) the following sequence is exact:

$$0 \longrightarrow W_h/\mathbb{R} \xrightarrow{\mathbf{grad}} \boldsymbol{Q}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h) \xrightarrow{\mathrm{div}} \S_h) \longrightarrow 0,$$

i.e., that the range of each of the operators in the sequence coincides with the nullspace of the following operator. It follows that if we define $\mathbf{grad}_h : S_h \to \boldsymbol{V}_h$ as the L^2 adjoint of the map $-\mathrm{div} : \boldsymbol{V}_h \to S_h$, and $\mathbf{curl}_h : \boldsymbol{V}_h \to \boldsymbol{Q}_h$ as the L^2 adjoint of $\mathbf{curl} : \boldsymbol{Q}_h \to \boldsymbol{V}_h$, then we have the two orthogonal decompositions:

$$\boldsymbol{V}_h = \mathbf{curl}\,\boldsymbol{Q}_h \oplus \mathbf{grad}_h S_h, \quad \boldsymbol{Q}_h = \mathbf{curl}_h \boldsymbol{V}_h \oplus \mathbf{grad}\, W_h.$$

These *discrete Helmholtz decompositions* are orthogonal in $\boldsymbol{L}^2$ and in $\boldsymbol{H}(\mathrm{div})$ for the first and $\boldsymbol{H}(\mathbf{curl})$ for the second.

Remark 2.1 If we were to consider the case where essential boundary conditions are imposed, the appropriate spaces would be $\mathring{W}_h = W_h \cap \mathring{H}^1$, $\mathring{\boldsymbol{Q}}_h = \boldsymbol{Q}_h \cap \mathring{\boldsymbol{H}}(\mathbf{curl})$, and $\mathring{\boldsymbol{V}}_h = \boldsymbol{V}_h \cap \mathring{\boldsymbol{H}}(\mathrm{div})$, and the corresponding exact sequence would be

$$0 \longrightarrow \mathring{W}_h \xrightarrow{\mathbf{grad}} \mathring{\boldsymbol{Q}}_h \xrightarrow{\mathbf{curl}} \mathring{\boldsymbol{V}}_h \xrightarrow{\mathrm{div}} S_h/\mathbb{R} \longrightarrow 0.$$

3. Abstract multigrid convergence

In this section, we recall some standard results for multigrid iterations, which will be the starting point for our analysis of multigrid methods for the linear systems of equations discussed in the previous section. Suppose

$$X_1 \subset X_2 \subset \cdots \subset X_J = X$$

is a sequence of finite dimensional subspaces of a Hilbert space, Y, and $\Lambda : X \times X \to \mathbb{R}$ is a symmetric positive definite bilinear form. For each j, we define $\Lambda_j : X_j \to X_j$ by $(\Lambda_j x, y) = \Lambda(x, y)$ for all $x, y \in X_j$. Our goal is the construction of an efficient multigrid iteration to solve or precondition equations of the form $\Lambda_J x = f$. Let $M_j : X \to X_j$ denote the Y-orthogonal projection, $P_j : X \to X_j$ the Λ-orthogonal projection, and $R_j : X_j \to X_j$ a linear operator (the smoother). For each j, we define an Y-symmetric operator $\Theta_j : X_j \to X_j$ by the standard multigrid V-cycle recursion with $m \geq 1$ smoothings. That is, we set $\Theta_1 = \Lambda_1^{-1}$ and for $j > 1$ and $f \in X_j$, we define $\Theta_j f = y_{2m+1}$ where

$$\begin{aligned}
&y_0 = 0 \in X_j,\\
&y_i = y_{i-1} + R_j(f - \Lambda_j y_{i-1}),\ i = 1, 2, \ldots, m,\\
&y_{m+1} = y_m + \Theta_{j-1} M_{j-1}(f - \Lambda_j y_m),\\
&y_i = y_{i-1} + R_j(f - \Lambda_j y_{i-1}),\ i = m+2, m+3, \ldots, 2m+1.
\end{aligned}$$

Then Θ_J is the V-cycle preconditioner for Λ_J. The following theorem gives conditions on the smoothers R_j which ensure convergence of the multigrid V-cycle (cf., [4], [5, Theorem 3.6], or [2, Theorem 5.1]).

Theorem 3.1 *Suppose that for each $j = 1, 2, \dots, J$, the smoother R_j is Y-symmetric and positive semidefinite and satisfies the conditions*

$$\Lambda([I - R_j \Lambda_j]x, x) \geq 0, \qquad x \in X_j,$$

and

$$(R_j^{-1}x, x) \leq \alpha \Lambda(x, x), \qquad x \in (I - P_{j-1})X_j,$$

where α is some constant. Then

$$0 \leq \Lambda([I - \Theta_J \Lambda_J]x, x) \leq \delta \Lambda(x, x), \qquad x \in X,$$

where $\delta = \alpha/(\alpha + 2m)$.

Hence, the multigrid error operator $I - \Theta_J \Lambda_J$ is a positive definite contraction with norm at most $\delta < 1$ independent of J and decreasing in m, and the preconditioned operator $\Theta_J \Lambda_J$ has eigenvalues between $1 - \delta$ and 1.

To obtain smoothers which satisfy the conditions of Theorem 3.1, we consider additive and multiplicative Schwarz operators. To describe these, we assume that for each j, there are spaces $X_j^k \subset X_j$ such that each $x \in X_j$ can be written in the form $\sum_{k=1}^K x^k$, with $x^k \in X_j^k$. Letting P_j^k denote the Λ-projection operator onto the space X_j^k, we can then define the unscaled additive Schwarz smoother by $R_j^{\mathrm{a}} = \sum_{k=1}^K P_j^k \Lambda^{-1}$ and then the smoother $R_j = \eta R_j^{\mathrm{a}}$, where η is a scaling factor. We also denote by R_j^{m} the usual multiplicative Schwarz smoother associated with the spaces X_j^k, i.e., for $f \in X_j$, $R_j^{\mathrm{m}} f := x^{2K}$, where

$$\begin{aligned}
&x^0 = 0,\\
&x^k = x^{k-1} - P_j^k(x^{k-1} - \Lambda_j^{-1}f), \quad k = 1, \dots, K,\\
&x^k = x^{k-1} - P_j^{2K+1-k}(x^{k-1} - \Lambda_j^{-1}f), \quad k = K+1, \dots, 2K.
\end{aligned}$$

The following theorem gives conditions on the decompositions of the X_j under which the Schwarz smoothers lead to a convergent multigrid iteration.

Theorem 3.2 *Suppose that*

$$\sum_{k=1}^K \sum_{l=1}^K \left| \Lambda(x^k, y^l) \right| \leq \beta \left[\sum_{k=1}^K \Lambda(x^k, x^k) \right]^{1/2} \left[\sum_{l=1}^K \Lambda(y^l, y^l) \right]^{1/2}, \quad x^k \in X_j^k,\ y^l \in X_j^l, \tag{3.1}$$

and

$$\text{(3.2)} \qquad \inf_{\substack{x^k \in X_j^k \\ x=\sum x^k}} \sum_{k=1}^{K} \Lambda(x^k, x^k) \le \gamma \Lambda(x,x), \qquad x \in (I - P_{j-1})X_j,$$

for some constants $\beta > 0$, $\gamma > 0$. *Then,*

- *(i) If* $\eta \le 1/\beta$, *the scaled additive smoothers* $R_j = \eta R_j^{\mathrm{a}}$ *satisfy the hypotheses of Theorem 3.1 with* $\alpha = \gamma/\eta$.
- *(ii) The multiplicative smoothers* $R_j = R_j^{\mathrm{m}}$ *satisfy the hypotheses of Theorem 3.1 with* $\alpha = \beta^2\gamma$.

Results of this type can be found in many places, for example in [5, Chapters 3 and 5], [7], [16, Chapter 5], and [18]. Therefore we merely sketch a proof here. The first hypothesis of Theorem 3.1 for the additive smoother follows from 3.1 with $x^k = y^k = P_j^k x$ and Schwarz's inequality. It is well known that the left hand side of 3.2 is precisely equal to $(R_j^{\mathrm{a}-1} x, x)$ (cf. equation (2.1) of [2]). The second hypothesis of Theorem 3.1 follows directly for the additive smoother. For the multiplicative smoother, the first hypothesis follows from the identity $\Lambda([I - R_j\Lambda_j]x, x) = \Lambda(Ex, Ex)$ where $E = (I - P_j^K)(I - P_j^{K-1}) \cdots (I - P_j^1)$. The second hypothesis is a consequence of the inequality $(R_j^{\mathrm{a}} x, x) \le \beta^2 (R_j^{\mathrm{m}} x, x)$, which is just Corollary 4.3 of [2], using the argument given at the end of Sect.5 of that paper.

4. Multigrid convergence in $\boldsymbol{H}(\mathbf{div})$ and $\boldsymbol{H}(\mathbf{curl})$

We consider a nested sequence of quasi-uniform tetrahedral meshes $\mathcal{T}_j$, $1 \le j \le J$. These give rise to spaces W_j, $\boldsymbol{Q}_j$, $\boldsymbol{V}_j$, and S_j and operators $\boldsymbol{\Lambda}_j^{\mathrm{d}} : \boldsymbol{V}_j \to \boldsymbol{V}_j$ and $\boldsymbol{\Lambda}_J^{\mathrm{c}} : \boldsymbol{Q}_j \to \boldsymbol{Q}_j$. In this section, we use Theorem 3.2 to obtain a convergence result for the multigrid V-cycle applied to the equation $\boldsymbol{\Lambda}_J^{\mathrm{d}} \boldsymbol{u} = \boldsymbol{f}$ or $\boldsymbol{\Lambda}_J^{\mathrm{c}} \boldsymbol{p} = \boldsymbol{g}$ in the space $X = \boldsymbol{V}_J$ or $\boldsymbol{Q}_J$. For the enclosing Hilbert space Y we take $\boldsymbol{L}^2$. We note that properties 3.1 and 3.2 only involve subspaces at two levels. Let h denote the mesh size of some mesh $\mathcal{T}_j$ and let H denote the mesh size of the next coarser mesh $\mathcal{T}_{j-1}$. To simplify notation, we shall write $\mathcal{T}_h$ and $\mathcal{T}_H$ for $\mathcal{T}_j$ and $\mathcal{T}_{j-1}$, and similarly in other cases where the subscripts j and $j-1$ arise.

To define the Schwarz smoothers, we must decompose the space $\boldsymbol{V}_h$ or $\boldsymbol{Q}_h$. For the space $\boldsymbol{V}_h$, three possible decompositions, based on face patches, edge patches, and vertex patches, are given in 2.1. From the point of view of implementation of the corresponding Schwarz smoother, the face-based decomposition, which has only two elements per patch, is most efficient, the edge-based less efficient, and the vertex-based Schwarz smoother the least efficient. However, as our theory will suggest and numerical computations in analogous situations reinforce [6], the face-based Schwarz smoother does not lead to an efficient multigrid algorithm. Below we shall prove that both decompositions

$$\boldsymbol{V}_h = \sum_{e\in\mathcal{E}_h} \boldsymbol{V}_h^e \qquad \text{and} \qquad \boldsymbol{V}_h = \sum_{v\in\mathcal{V}_h} \boldsymbol{V}_h^v, \tag{4.1}$$

yield Schwarz smoothers that satisfy the conditions of Theorem 3.2 with constants independent of h and κ. In [11] Hiptmair generalizes to three dimension a decomposition used in two dimensions by Vassilevski and Wang [17], namely,

$$\boldsymbol{V}_h = \sum_{f\in\mathcal{F}_h} \boldsymbol{V}_h^f + \sum_{e\in\mathcal{E}_h} \mathbf{curl}\,\boldsymbol{Q}_h^e. \tag{4.2}$$

The implementation of the corresponding smoother, which may be more efficient than the smoother based on edge patches, is discussed in [11]. Our analysis below applies to this smoother as well.

For the space $\boldsymbol{Q}_h$ we may use either the decomposition

$$\boldsymbol{Q}_h = \sum_{v\in\mathcal{V}_h} \boldsymbol{Q}_h^v, \tag{4.3}$$

or one due to Hiptmair [12],

$$\boldsymbol{Q}_h = \sum_{e\in\mathcal{E}_h} \boldsymbol{Q}_h^e + \sum_{v\in\mathcal{V}_h} \mathbf{grad}\, W_h^v. \tag{4.4}$$

It is easy to check that since no point belongs to more than six of the Ω_h^e or four of the Ω_h^v or Ω_h^f, all these decompositions satisfy the condition 3.1 with β independent of h, ρ, and κ (β will never exceed 10). It thus only remains to verify condition 3.2, which we state for the particular case of the first smoother in 4.1 and the smoother in 4.3 in the following two theorems. The verification for the other smoothers will be remarked on below. For these theorems (only) we require the bounded refinement hypothesis $H \le ch$. (In practice, values of c around 2 are common.)

Theorem 4.1 *Assume that $H \le ch$ and that $\boldsymbol{v} \in (\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\boldsymbol{V}_h$ be given. There exists a decomposition $\boldsymbol{v} = \sum_{e\in\mathcal{E}_h} \boldsymbol{v}^e$, where $\boldsymbol{v}^e \in \boldsymbol{V}_h^e$, and a constant γ depending on c but independent of h, ρ, and κ such that*

$$\sum_{e\in\mathcal{E}_h} \Lambda^{\mathrm{d}}(\boldsymbol{v}^e, \boldsymbol{v}^e) \le \gamma \Lambda^{\mathrm{d}}(\boldsymbol{v}, \boldsymbol{v}).$$

Theorem 4.2 *Assume that* $H \le ch$ *and that* $\boldsymbol{q} \in (\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{c}})\boldsymbol{Q}_h$ *be given. There exists a decomposition* $\boldsymbol{q} = \sum_{v\in\mathcal{V}_h} \boldsymbol{q}^v$, *where* $\boldsymbol{q}^v \in \boldsymbol{Q}_h^v$, *and a constant* γ *depending on* c *but independent of* h, ρ, *and* κ *such that*

$$\sum_{v\in\mathcal{V}_h} \Lambda^{\mathrm{c}}(\boldsymbol{q}^v, \boldsymbol{q}^v) \le \gamma \Lambda^{\mathrm{c}}(\boldsymbol{q}, \boldsymbol{q}).$$

To prove these results, we will make use of the discrete Helmholtz decompositions described in Sect.2. For these decompositions, the following two propositions, for $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl}\,)$, respectively, will be the key ingredients of the analysis. The proofs of these propositions, unlike the proof of Theorems 4.1 and 4.2, do not require that $H \le ch$. Also, since Theorems 4.1 and 4.2 are unaffected by scaling of the bilinear form, in the remainder of the paper we assume, without loss of generality, that $\rho = 1$.

Proposition 4.3 *Suppose that* $\boldsymbol{u} \in \boldsymbol{V}_h$ *and that* $\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u} \in \boldsymbol{V}_h$ *has the discrete Helmholtz decomposition*

$$\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u} = \mathbf{grad}\,_h s_h + \mathbf{curl}\,\boldsymbol{q}_h,$$

for some $s_h \in S_h$ *and* $\boldsymbol{q}_h \in \mathbf{curl}\,_h \boldsymbol{V}_h$. *Then*

$$\kappa\|\mathbf{grad}\,_h s_h\| \le cH\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|_{\Lambda^{\mathrm{d}}}, \qquad \|\boldsymbol{q}_h\| \le cH\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|.$$

Proposition 4.4 *Suppose that* $\boldsymbol{p} \in \boldsymbol{Q}_h$ *and that* $\boldsymbol{p} - \boldsymbol{P}_H^{\mathrm{c}}\boldsymbol{p} \in \boldsymbol{Q}_h$ *has the discrete Helmholtz decomposition*

$$\boldsymbol{p} - \boldsymbol{P}_H^{\mathrm{c}}\boldsymbol{p} = \mathbf{grad}\, w_h + \mathbf{curl}\,_h \boldsymbol{v}_h,$$

for some $w_h \in W_h/\mathbb{R}$ *and* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *with* $\mathrm{div}\boldsymbol{v}_h = 0$. *Then*

$$\|w_h\| \le cH\|\boldsymbol{p} - \boldsymbol{P}_H^{\mathrm{c}}\boldsymbol{p}\|, \qquad \kappa\|\mathbf{curl}\,_h \boldsymbol{v}_h\| \le cH\|\boldsymbol{p} - \boldsymbol{P}_H^{\mathrm{c}}\boldsymbol{p}\|_{\Lambda^{\mathrm{c}}}.$$

The proof of these results requires a series of intermediate results and will be given in the next section. We now show how these propositions may be used to establish Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Since $\boldsymbol{v} \in (\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\boldsymbol{V}_h$, it follows from Proposition 4.3 and the bounded refinement hypothesis that $\boldsymbol{v}$ admits a discrete Helmholtz decomposition

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + \mathbf{curl}\,\boldsymbol{q},$$

where $\tilde{\boldsymbol{v}} \in \mathbf{grad}\,_h S_h$ and $\boldsymbol{q} \in \boldsymbol{Q}_h$ satisfy the bounds

$$\|\tilde{\boldsymbol{v}}\| \le \|\boldsymbol{v}\|, \quad \kappa\|\tilde{\boldsymbol{v}}\| \le ch\|\boldsymbol{v}\|_{\Lambda^{\mathrm{d}}}, \quad \|\boldsymbol{q}\| \le ch\|\boldsymbol{v}\|. \tag{4.5}$$

Following the discussion of Sect.2, we can write $\tilde{\boldsymbol{v}} = \sum_{e\in\mathcal{E}_h} \tilde{\boldsymbol{v}}^e$ and $\boldsymbol{q} = \sum_{e\in\mathcal{E}_h} \boldsymbol{q}^e$ with

$$\sum_{e\in\mathcal{E}_h} \|\tilde{\boldsymbol{v}}^e\|^2 \le c\|\tilde{\boldsymbol{v}}\|^2, \quad \sum_{e\in\mathcal{E}_h} \|\boldsymbol{q}^e\|^2 \le c\|\boldsymbol{q}\|^2. \tag{4.6}$$

Then $\boldsymbol{v} = \sum_{e\in\mathcal{E}_h} \boldsymbol{v}^e$ where $\boldsymbol{v}^e := \tilde{\boldsymbol{v}}^e + \mathbf{curl}\,\boldsymbol{q}^e$. Moreover, using an inverse inequality,

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} \|\boldsymbol{v}^e\|_{\Lambda^{\mathrm{d}}}^2 &= \sum_{e\in\mathcal{E}_h} (\|\tilde{\boldsymbol{v}}^e\|_{\Lambda^{\mathrm{d}}}^2 + \|\mathbf{curl}\,\boldsymbol{q}^e\|^2) \\ &\le c \sum_{e\in\mathcal{E}_h} [(1+\kappa^2 h^{-2})\|\tilde{\boldsymbol{v}}^e\|^2 + h^{-2}\|\boldsymbol{q}^e\|^2],\end{aligned} \tag{4.7}$$

and the theorem follows from 4.5–4.7. □

Proof of Theorem 4.2. Since $\boldsymbol{q} \in (\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{c}})\boldsymbol{Q}_h$, it follows from Proposition 4.4 and the bounded refinement hypothesis that $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = \tilde{\boldsymbol{q}} + \mathbf{grad}\, w,$$

where $\tilde{\boldsymbol{q}} \in \mathbf{curl}_h \boldsymbol{V}_h$ and $w \in W_h$ satisfy the estimates

$$\|\tilde{\boldsymbol{q}}\| \le \|\boldsymbol{q}\|, \quad \kappa\|\tilde{\boldsymbol{q}}\| \le ch\|\boldsymbol{q}\|_{\Lambda^{\mathrm{c}}}, \quad \|w\| \le ch\|\boldsymbol{q}\|.$$

Writing $\tilde{\boldsymbol{q}} = \sum_{v\in\mathcal{V}_h} \tilde{\boldsymbol{q}}^v$ and $w = \sum_{v\in\mathcal{V}_h} w^v$, and setting $\boldsymbol{q}^v = \tilde{\boldsymbol{q}}^v + \mathbf{grad}\, w^v$, we complete the proof as for the preceding theorem. □

Remark 4.1 The proof of Theorem 4.1 applies almost without modification if the decomposition $\boldsymbol{V}_h = \sum_{e\in\mathcal{E}_h} \boldsymbol{V}_h^e$ is replaced by either the second decomposition in 4.1 or the decomposition in 4.2. Similarly, the proof of Theorem 4.2 applies to the decomposition in 4.4 as well. It is also clear why we cannot use the face-based decomposition of $\boldsymbol{V}_h$ in Theorem 4.1, since the proof would require a corresponding face-based decomposition of $\boldsymbol{Q}_h$, which does not exist.

5. Two-level estimates for mixed finite elements

In this section we prove Propositions 4.3 and 4.4. Our proofs are based on estimates for the approximation of discretely irrotational vector fields in $\boldsymbol{V}_h$ and discretely solenoidal vector fields in $\boldsymbol{Q}_h$ by discretely irrotational and solenoidal fields in $\boldsymbol{V}_H$ and $\boldsymbol{Q}_H$, respectively. These two-level approximation results, in turn, rely on estimates for mixed finite element methods based on $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. We begin this section with a discussion of such methods.

For the $\boldsymbol{H}(\mathrm{div})$ case, let $f \in L^2$ and define $(s, \boldsymbol{v})$ as the unique critical point (a saddle) of

$$\mathcal{L}^{td}(s, \boldsymbol{v}) := \frac{1}{2}\|\boldsymbol{v}\|^2 + (\mathrm{div}\boldsymbol{v}, s) - (f, s)$$

over $L^2 \times \boldsymbol{H}(\mathrm{div})$. This is a mixed variational formulation of the Dirichlet boundary value problem

(5.1) $$\boldsymbol{v} = \mathbf{grad}\, s,\ \mathrm{div}\boldsymbol{v} = f \text{ in } \Omega, \quad s = 0 \text{ on } \partial\Omega.$$

The mixed finite element approximation $(s_h, \boldsymbol{v}_h)$ to $(s, \boldsymbol{v})$ is the unique critical point of $\mathcal{L}^{td}$ over $S_h \times \boldsymbol{V}_h$. It is determined by the equations $\boldsymbol{v}_h = \mathbf{grad}_h s_h$, $\mathrm{div}\boldsymbol{v}_h = \Pi_h^S f$, and $\boldsymbol{v}_h$ alone is characterized as the unique function in $\mathbf{grad}_h S_h$ satisfying the latter equation. A basic estimate for mixed methods is

(5.2) $$\|\boldsymbol{v} - \boldsymbol{v}_h\| \le \|\boldsymbol{v} - \boldsymbol{\Pi}_h^V \boldsymbol{v}\|, \qquad \boldsymbol{v} \in \boldsymbol{H}^1,$$

which is a consequence of the commutativity property $\mathrm{div}\boldsymbol{\Pi}_h^V = \Pi_h^S \mathrm{div}$. From the properties of the operator $\boldsymbol{\Pi}_h^V$ one also easily derives the inf–sup condition:

$$\inf_{s \in S_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\mathrm{div}\boldsymbol{v}, s)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div})}\|s\|} \ge \sigma > 0.$$

A useful consequence is the discrete Poincaré inequality:

(5.3) $$\|s\| \le \sigma^{-1}\|\mathbf{grad}_h s\|, \quad s \in S_h.$$

To describe the corresponding situation in the $\boldsymbol{H}(\mathbf{curl})$ case, we introduce the space

$$\begin{aligned}\boldsymbol{Z} &:= \{\, \boldsymbol{z} \in \boldsymbol{H}(\mathrm{div}) \mid \mathrm{div}\boldsymbol{z} = 0 \,\} \\ &= \boldsymbol{H}(\mathrm{div}) \cap (\mathbf{grad}\, \mathring{H}^1)^{\perp} = \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}),\end{aligned}$$

and the discrete analogue

$$\boldsymbol{Z}_h := \{\, \boldsymbol{z}_h \in \boldsymbol{V}_h \mid \mathrm{div}\boldsymbol{z}_h = 0 \,\} = \boldsymbol{V}_h \cap (\mathbf{grad}_h S_h)^{\perp} = \mathbf{curl}\, \boldsymbol{Q}_h.$$

The mixed variational problem we consider now begins with a function $\boldsymbol{f} \in \boldsymbol{Z}$ and characterizes $(\boldsymbol{z}, \boldsymbol{q}) \in \boldsymbol{Z} \times \boldsymbol{H}(\mathbf{curl})$ as the unique critical point (again a saddle) of

$$\mathcal{L}^{\mathrm{c}}(\boldsymbol{z}, \boldsymbol{q}) := \frac{1}{2}\|\boldsymbol{q}\|^2 - (\mathbf{curl}\, \boldsymbol{q}, \boldsymbol{z}) + (\boldsymbol{f}, \boldsymbol{z}).$$

This corresponds to the boundary value problem

(5.4) $$\boldsymbol{q} = \mathbf{curl}\, \boldsymbol{z},\ \mathbf{curl}\, \boldsymbol{q} = \boldsymbol{f},\ \mathrm{div}\boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega.$$

For this problem we have $\boldsymbol{q}, \boldsymbol{z} \in \boldsymbol{H}^1$ and

$$\|\boldsymbol{q}\|_1 \le c\|\boldsymbol{f}\|, \qquad \|\boldsymbol{z}\|_1 \le c\|\boldsymbol{q}\|. \tag{5.5}$$

Indeed, since the normal component of $\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{z}$ is the tangential divergence of $\boldsymbol{z} \times \boldsymbol{n}$, which vanishes on $\partial\Omega$, it follows that $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. The estimates on $\boldsymbol{q}$ and $\boldsymbol{z}$ are then given in Theorems 2.1 and 2.2 of [8], respectively. (These results depend on the assumption of convexity.)

The mixed finite element approximation $(\boldsymbol{z}_h, \boldsymbol{q}_h)$ is the unique critical point of $\mathcal{L}^c$ over $\boldsymbol{Z}_h \times \boldsymbol{Q}_h$. It is determined by the equations $\boldsymbol{q}_h = \mathbf{curl}_h \boldsymbol{z}_h$, $\mathbf{curl}\,\boldsymbol{q}_h = \boldsymbol{\Pi}_h^Z \boldsymbol{f}$, where $\boldsymbol{\Pi}_h^Z : \boldsymbol{L}^2 \to \boldsymbol{Z}_h$ is the $\boldsymbol{L}^2$ projection, and $\boldsymbol{q}_h$ alone is characterized as the unique function in $\mathbf{curl}_h \boldsymbol{V}_h$ satisfying the latter equation. At this point, an essential difference between the mixed approximation of 5.4 and 5.1 arises. It is *not* true that $\mathbf{curl}\,\boldsymbol{\Pi}_h^Q \boldsymbol{q} = \boldsymbol{\Pi}_h^Z \mathbf{curl}\,\boldsymbol{q}$ for all smooth functions $\boldsymbol{q}$ (since $\boldsymbol{\Pi}_h^Z$ does not coincide with $\boldsymbol{\Pi}_h^V$, even when applied to irrotational fields). As a result, it is not in general true that $\|\boldsymbol{q} - \boldsymbol{q}_h\| \le \|\boldsymbol{q} - \boldsymbol{\Pi}_h^Q \boldsymbol{q}\|$. However, this estimate is true in the special case that $\boldsymbol{f} \in \boldsymbol{Z}_h$, i.e.,

$$\|\boldsymbol{q} - \boldsymbol{q}_h\| \le \|\boldsymbol{q} - \boldsymbol{\Pi}_h^Q \boldsymbol{q}\|, \qquad \boldsymbol{q} \in \boldsymbol{H}^1 \text{ such that } \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{V}_h. \tag{5.6}$$

Indeed, in this case

$$\mathbf{curl}\,\boldsymbol{\Pi}_h^Q \boldsymbol{q} = \boldsymbol{\Pi}_h^V \mathbf{curl}\,\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{q} = \boldsymbol{\Pi}_h^Z \mathbf{curl}\,\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{q}_h,$$

so $\boldsymbol{\Pi}_h^Q \boldsymbol{q} - \boldsymbol{q}_h$ is curl-free. It then follows directly from the defining equations of the mixed method that $(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\Pi}_h^Q \boldsymbol{q} - \boldsymbol{q}_h) = 0$, which gives 5.6.

Notice that the hypothesis $\mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{V}_h$ is also what is needed for the approximation estimate 2.4. Combining 5.6, 2.4, and the continuous inf–sup condition, we get the discrete inf–sup condition,

$$\inf_{\boldsymbol{z} \in \boldsymbol{Z}_h} \sup_{\boldsymbol{q} \in \boldsymbol{Q}_h} \frac{(\mathbf{curl}\,\boldsymbol{q}, \boldsymbol{z})}{\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl})} \|\boldsymbol{z}\|} \ge \sigma > 0,$$

in the usual way. This in turn implies an analogue of the discrete Poincaré inequality,

$$\|\boldsymbol{z}\| \le \sigma^{-1} \|\mathbf{curl}_h \boldsymbol{z}\|, \quad \boldsymbol{z} \in \boldsymbol{Z}_h. \tag{5.7}$$

Having completed the necessary discusion of mixed methods, we now turn to the key *two-level error estimates*, from which Propositions 4.3 and 4.4 will follow. Given a finite element vector field with respect to some fine mesh, these estimates give bounds for the approximation to it obtained using mixed finite elements on a coarser mesh. Discrete norms compensate for the lack of regularity of the fine mesh solution.

Lemma 5.1 *Given* $\boldsymbol{v}_h \in \mathbf{grad}_h S_h$*, let* $\boldsymbol{v}_H$ *be the unique element of* $\mathbf{grad}_H S_H$ *satisfying* $\operatorname{div}\boldsymbol{v}_H = \Pi_H^S \operatorname{div}\boldsymbol{v}_h$*. Then*

$$\|\boldsymbol{v}_h - \boldsymbol{v}_H\| \le cH\|\operatorname{div}\boldsymbol{v}_h\|, \quad \|\operatorname{div}(\boldsymbol{v}_h - \boldsymbol{v}_H)\| \le cH\|\mathbf{grad}_h \operatorname{div}\boldsymbol{v}_h\|.$$

Lemma 5.2 *Given* $\boldsymbol{q}_h \in \mathbf{curl}_h \boldsymbol{V}_h$*, let* $\boldsymbol{q}_H$ *be the unique element of* $\mathbf{curl}_H \boldsymbol{V}_H$ *satisfying* $\mathbf{curl}\,\boldsymbol{q}_H = \boldsymbol{\Pi}_H^Z \mathbf{curl}\,\boldsymbol{q}_h$*. Then*

$$\|\boldsymbol{q}_h - \boldsymbol{q}_H\| \le cH\|\mathbf{curl}\,\boldsymbol{q}_h\|, \quad \|\mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H)\| \le cH\|\mathbf{curl}_h \mathbf{curl}\,\boldsymbol{q}_h\|.$$

We shall first prove Lemma 5.1. Combining it with an appropriate duality argument, we establish Proposition 4.3. We shall then prove Lemma 5.2 and Proposition 4.4.

Proof of Lemma 5.1. Define $(\boldsymbol{v}, s)$ from 5.1 with $\boldsymbol{f} = \operatorname{div}\boldsymbol{v}_h$. Then $\boldsymbol{v}_h$ and $\boldsymbol{v}_H$ are the mixed approximations to $\boldsymbol{v}$ in $\boldsymbol{V}_h$ and $\boldsymbol{V}_H$, respectively. Applying 5.2, 2.3, and 2-regularity for the Dirichlet problem on a convex polyhedron, we obtain

$$\|\boldsymbol{v} - \boldsymbol{v}_H\| \le \|\boldsymbol{v} - \boldsymbol{\Pi}_H^V \boldsymbol{v}\| \le cH\|\boldsymbol{v}\|_1 \le cH\|\operatorname{div}\boldsymbol{v}_h\|,$$

and, similarly, $\|\boldsymbol{v} - \boldsymbol{v}_h\| \le ch\|\operatorname{div}\boldsymbol{v}_h\|$. The first estimate thus follows from the triangle inequality.

Next we prove that for any $r_h \in S_h$,

$$\|r_h - \Pi_H^S r_h\| \le cH\|\mathbf{grad}_h r_h\|. \tag{5.8}$$

In particular, we may take $r_h = \operatorname{div}\boldsymbol{v}_h$ in this estimate, to get

$$\|\operatorname{div}\boldsymbol{v}_h - \operatorname{div}\boldsymbol{v}_H\| \le cH\|\mathbf{grad}_h \operatorname{div}\boldsymbol{v}_h\|.$$

To prove 5.8, we define a function $\boldsymbol{u}$ which satisfies

$$\operatorname{div}\boldsymbol{u} = r_h - \Pi_H^S r_h, \qquad \|\boldsymbol{u}\|_1 \le \|r_h - \Pi_H^S r_h\|.$$

Then

$$\begin{aligned}
\|r_h - \Pi_H^S r_h\|^2 &= (\operatorname{div}\boldsymbol{u}, r_h - \Pi_H^S r_h) = (\operatorname{div}\boldsymbol{u}, \Pi_h^S r_h - \Pi_H^S r_h) \\
&= ([\Pi_h^S - \Pi_H^S]\operatorname{div}\boldsymbol{u}, r_h) = (\operatorname{div}[\boldsymbol{\Pi}_h^V - \boldsymbol{\Pi}_H^V]\boldsymbol{u}, r_h) \\
&= ([\boldsymbol{\Pi}_h^V - \boldsymbol{\Pi}_H^V]\boldsymbol{u}, \mathbf{grad}_h r_h) \\
&\le (\|\boldsymbol{\Pi}_h^V \boldsymbol{u} - \boldsymbol{u}\| + \|\boldsymbol{u} - \boldsymbol{\Pi}_H^V \boldsymbol{u}\|)\|\mathbf{grad}_h r_h\| \\
&\le cH\|\boldsymbol{u}\|_1\|\mathbf{grad}_h r_h\| \le cH\|r_h - \Pi_H^S r_h\|\|\mathbf{grad}_h r_h\|,
\end{aligned}$$

which implies 5.8. □

Proof of Proposition 4.3. The proposition directly generalizes the corresponding two-dimensional result, Lemma 3.1 of [2]. The proof of the bound on $\mathbf{grad}_h s_h$ is entirely analogous to the argument in [2], but the bound for $\boldsymbol{q}_h$ requires the use of a more complicated duality argument. First, observe that

$$(5.9)\qquad (\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{curl}\,\boldsymbol{r}) = \Lambda^{\mathrm{d}}(\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{r}) = 0, \qquad \boldsymbol{r} \in \boldsymbol{Q}_H.$$

Define $(\boldsymbol{q}, \boldsymbol{z})$ as in 5.4 with $\boldsymbol{f}$ replaced by $\mathbf{curl}\,\boldsymbol{q}_h$. Then $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ is the mixed approximation to $\boldsymbol{q}$, and hence, by 5.6, 2.4, and 5.5,

$$(5.10)\qquad \|\boldsymbol{q} - \boldsymbol{q}_h\| \le \|\boldsymbol{q} - \boldsymbol{\Pi}_h^Q \boldsymbol{q}\| \le ch\|\boldsymbol{q}\|_1 \le ch\|\mathbf{curl}\,\boldsymbol{q}_h\|.$$

Since $\mathrm{div}\boldsymbol{z} = 0$, $\mathrm{div}\boldsymbol{\Pi}_H^V\boldsymbol{z} = 0$, and so $\boldsymbol{\Pi}_H^V\boldsymbol{z} \in \mathbf{curl}\,\boldsymbol{Q}_H$. We may therefore apply 5.9, 2.3, and 5.5 to obtain

$$\begin{aligned}\|\boldsymbol{q}\|^2 &= (\boldsymbol{q}, \mathbf{curl}\,\boldsymbol{z}) = (\mathbf{curl}\,\boldsymbol{q}, \boldsymbol{z}) = (\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{z})\\ &= (\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{z} - \boldsymbol{\Pi}_H^V\boldsymbol{z}) \le cH\|\boldsymbol{z}\|_1\|\mathbf{curl}\,\boldsymbol{q}_h\| \le cH\|\boldsymbol{q}\|\|\mathbf{curl}\,\boldsymbol{q}_h\|.\end{aligned}$$

Hence, $\|\boldsymbol{q}\| \le cH\|\mathbf{curl}\,\boldsymbol{q}_h\|$. Combining this with 5.10, we obtain

$$\|\boldsymbol{q}_h\| \le cH\|\mathbf{curl}\,\boldsymbol{q}_h\| \le cH\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|.$$

This completes the proof of the second estimate of the proposition.

Since the first estimate is vacuous if $\kappa = 0$, we assume $\kappa > 0$. Since $\boldsymbol{\Lambda}_h^{\mathrm{d}}$ maps $\mathbf{grad}_h S_h$ onto itself, we have $\boldsymbol{v}_h = (\boldsymbol{\Lambda}_h^{\mathrm{d}})^{-1}\mathbf{grad}_h s_h \in \mathbf{grad}_h S_h$. Defining $\boldsymbol{v}_H \in \boldsymbol{V}_H$ as in Lemma 5.1, we have

$$\begin{aligned}\|\boldsymbol{v}_h - \boldsymbol{v}_H\|_{\Lambda_h^{\mathrm{d}}}^2 &\le cH^2(\|\mathrm{div}\boldsymbol{v}_h\|^2 + \kappa^2\|\mathbf{grad}_h\mathrm{div}\boldsymbol{v}_h\|^2)\\ &\le cH^2\kappa^{-2}(\|\boldsymbol{v}_h\|^2 + 2\kappa^2\|\mathrm{div}\boldsymbol{v}_h\|^2 + \kappa^4\|\mathbf{grad}_h\mathrm{div}\boldsymbol{v}_h\|^2)\\ &= cH^2\kappa^{-2}\|\boldsymbol{\Lambda}_h^{\mathrm{d}}\boldsymbol{v}_h\|^2 = cH^2\kappa^{-2}\|\mathbf{grad}_h s_h\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathbf{grad}_h s_h\|^2 &= \Lambda^{\mathrm{d}}(\mathbf{grad}_h s_h, \boldsymbol{v}_h) = \Lambda^{\mathrm{d}}(\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}, \boldsymbol{v}_h)\\ &= \Lambda^{\mathrm{d}}(\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v}_H) \le \|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|_{\Lambda^{\mathrm{d}}}\|\boldsymbol{v}_h - \boldsymbol{v}_H\|_{\Lambda^{\mathrm{d}}}\\ &\le cH\kappa^{-1}\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|_{\Lambda^{\mathrm{d}}}\|\mathbf{grad}_h s_h\|. \square\end{aligned}$$

We now prove Lemma 5.2, and then Proposition 4.4, which will follow easily. The proof of the lemma is substantially more involved than that of Lemma 5.1, because the error estimate $\|\boldsymbol{q} - \boldsymbol{q}_H\| \le \|\boldsymbol{q} - \boldsymbol{\Pi}_H^Q\boldsymbol{q}\|$ is not valid (reflecting the lack of the commutativity propery $\mathbf{curl}\,\boldsymbol{\Pi}_h^Q = \boldsymbol{\Pi}_h^Z\mathbf{curl}$).

Proof of Lemma 5.2. The lemma does not involve the parameter κ. So as not to introduce additional notation, the notation Λ^{d} is used in this proof to denote the unweighted inner product in $\boldsymbol{H}(\mathrm{div})$ ($\kappa = 1$), and $\boldsymbol{P}_H^{\mathrm{d}}$ is used to denote the corresponding orthogonal projection.

Since $\mathbf{curl}\,\boldsymbol{q}_H = \boldsymbol{\Pi}_H^Z \mathbf{curl}\,\boldsymbol{q}_h$ where $\boldsymbol{\Pi}_H^Z$ is the $\boldsymbol{L}^2$ projection onto $\boldsymbol{Z}_H$, we obviously have

$$\|\mathbf{curl}\,\boldsymbol{q}_H\| \le c\|\mathbf{curl}\,\boldsymbol{q}_h\|. \tag{5.11}$$

Define $(\boldsymbol{q}, \boldsymbol{z})$ by the boundary value problem 5.4 with $\boldsymbol{f}$ replaced by $\mathbf{curl}\,\boldsymbol{q}_h$. Since $\boldsymbol{q}_h$ is the mixed approximation of $\boldsymbol{q}$ in $\boldsymbol{Q}_h$ and $\mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{V}_h$, we are able to use 5.6 to estimate $\boldsymbol{q} - \boldsymbol{q}_h$. While $\boldsymbol{q}_H$ is the mixed approximation of $\boldsymbol{q}$ in $\boldsymbol{Q}_H$, it is not true that $\mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{V}_H$, so we cannot estimate $\boldsymbol{q} - \boldsymbol{q}_H$ in the same way. Therefore we define $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{z}})$ by 5.4 with $\boldsymbol{f}$ replaced by $\mathbf{curl}\,\boldsymbol{q}_H$. (The analogous complication did not arise in the proof of Lemma 5.1.) Setting $\boldsymbol{\phi} = \boldsymbol{q} - \bar{\boldsymbol{q}}$ and $\boldsymbol{\psi} = \boldsymbol{z} - \bar{\boldsymbol{z}}$, we obtain

$$\begin{gathered}
\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{\phi}, \quad \|\boldsymbol{\psi}\|_1 \le c\|\boldsymbol{\phi}\|, \\
\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{curl}\,(\boldsymbol{q} - \bar{\boldsymbol{q}}) = \mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H), \\
\|\boldsymbol{\phi}\|_1 \le c\|\mathbf{curl}\,\boldsymbol{q}_h\| + c\|\mathbf{curl}\,\boldsymbol{q}_H\| \le c\|\mathbf{curl}\,\boldsymbol{q}_h\|, \\
\|\boldsymbol{\phi} - (\boldsymbol{q}_h - \boldsymbol{q}_H)\| \le \|\boldsymbol{q} - \boldsymbol{q}_h\| + \|\bar{\boldsymbol{q}} - \boldsymbol{q}_H\| \le cH\|\mathbf{curl}\,\boldsymbol{q}_h\|,
\end{gathered}$$

where in the last estimate we have used 5.6, 2.4, and 5.5 twice, and then 5.11.

We estimate $\|\boldsymbol{\phi}\|$ using the same duality argument we used to estimate $\|\boldsymbol{q}\|$ in the proof of Proposition 4.3. Since $\boldsymbol{\Pi}_H^V \boldsymbol{\psi} \in \boldsymbol{Z}_H$ (which follows from the commutativity relation $\operatorname{div} \boldsymbol{\Pi}_H^V = \boldsymbol{\Pi}_H^S \operatorname{div}$), and $\mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H) \perp \boldsymbol{Z}_H$, we find

$$\begin{aligned}
\|\boldsymbol{\phi}\|^2 &= (\boldsymbol{\phi}, \mathbf{curl}\,\boldsymbol{\psi}) = (\mathbf{curl}\,\boldsymbol{\phi}, \boldsymbol{\psi}) = (\mathbf{curl}\,[\boldsymbol{q}_h - \boldsymbol{q}_H], \boldsymbol{\psi}) \\
&= (\mathbf{curl}\,[\boldsymbol{q}_h - \boldsymbol{q}_H], \boldsymbol{\psi} - \boldsymbol{\Pi}_H^V \boldsymbol{\psi}) \le cH\|\mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H)\|\|\boldsymbol{\psi}\|_1 \\
&\le cH\|\mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H)\|\|\boldsymbol{\phi}\|.
\end{aligned}$$

This implies that $\|\boldsymbol{\phi}\| \le cH\|\mathbf{curl}\,(\boldsymbol{q}_h - \boldsymbol{q}_H)\| \le CH\|\mathbf{curl}\,\boldsymbol{q}_h\|$, and so we obtain the first estimate of the lemma.

It remains to prove the second estimate. For this estimate, too, we cannot simply use the analogue of the argument that established the second estimate of Lemma 5.1. This time the problem can be traced to the failure of the commutativity property $\boldsymbol{\Pi}_H^Z \mathbf{curl} = \mathbf{curl}\,\boldsymbol{\Pi}_H^Q$, even though the analogous property $\boldsymbol{\Pi}_H^S \operatorname{div} = \operatorname{div} \boldsymbol{\Pi}_H^V$ is valid. Instead we shall derive the estimate by establishing the following three facts:

$$\mathbf{curl}\,\boldsymbol{q}_h - \mathbf{curl}\,\boldsymbol{q}_H = (\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\mathbf{curl}\,\boldsymbol{q}_h + \mathbf{grad}_H s_H, \quad \text{for some } s_H \in S_H, \tag{5.12}$$

$$\|\mathbf{grad}_H s_H\| \le c\|(\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\mathbf{curl}\,\boldsymbol{q}_h\|, \tag{5.13}$$

$$\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}} \boldsymbol{u}\| \le cH\|\mathbf{curl}_h \boldsymbol{u}\|, \qquad \boldsymbol{u} \in \mathbf{curl}\,\boldsymbol{Q}_h. \tag{5.14}$$

The desired estimate follows by taking $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{q}_h$ in 5.14 and using 5.12 and 5.13.

The first statement follows from the equations

$$
\begin{aligned}
(\mathbf{curl}\,\boldsymbol{q}_H, \mathbf{curl}\,\boldsymbol{r}_H) &= (\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{curl}\,\boldsymbol{r}_H) = \Lambda^{\mathrm{d}}(\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{curl}\,\boldsymbol{r}_H) \\
&= \Lambda^{\mathrm{d}}(\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{curl}\,\boldsymbol{r}_H) \\
&= (\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{curl}\,\boldsymbol{r}_H), \qquad \boldsymbol{r}_H \in \boldsymbol{Q}_H.
\end{aligned}
$$

To prove 5.13, we use the Helmholtz decomposition of $\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h$ and the definition of $\boldsymbol{P}_H^{\mathrm{d}}$ to see that for any $\boldsymbol{v}_H \in \boldsymbol{V}_H$,

$$
\begin{aligned}
(\mathrm{div}\mathbf{grad}_H s_H, \mathrm{div}\boldsymbol{v}_H) &= (\mathrm{div}\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \mathrm{div}\boldsymbol{v}_H) \\
&= \Lambda^{\mathrm{d}}(\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{v}_H) - (\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{v}_H) \\
&= (\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{v}_H) - (\boldsymbol{P}_H^{\mathrm{d}}\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{v}_H) \\
&= ([\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}}]\mathbf{curl}\,\boldsymbol{q}_h, \boldsymbol{v}_H).
\end{aligned} \tag{5.15}
$$

Now

$$
\begin{aligned}
\|\mathbf{grad}_H s_H\|^2 = -(\mathrm{div}\mathbf{grad}_H s_H, s_H) &\le \|\mathrm{div}\mathbf{grad}_H s_H\|\|s_H\| \\
&\le c\|\mathrm{div}\mathbf{grad}_H s_H\|\|\mathbf{grad}_H s_H\|,
\end{aligned}
$$

by the discrete Poincaré inequality 5.3. Thus

$$\|\mathbf{grad}_H s_H\| \le c\|\mathrm{div}\mathbf{grad}_H s_H\|,$$

and taking $\boldsymbol{v}_H = \mathbf{grad}_H s_H$ in 5.15, we get

$$
\begin{aligned}
\|\mathbf{grad}_H s_H\|^2 &\le c\|\mathrm{div}\mathbf{grad}_H s_H\|^2 \\
&= c([\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}}]\mathbf{curl}\,\boldsymbol{q}_h, \mathbf{grad}_H s_H) \\
&\le c\|(\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\mathbf{curl}\,\boldsymbol{q}_h\|\|\mathbf{grad}_H s_H\|,
\end{aligned}
$$

as desired.

It remains to prove 5.14. For $\boldsymbol{u} \in \mathbf{curl}\,\boldsymbol{Q}_h$, we use the discrete Helmholtz decomposition to write

$$(\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{p} + \mathbf{grad}_h s, \quad s \in S_h, \quad \boldsymbol{p} \in \mathbf{curl}_h \boldsymbol{V}_h,$$

and then to write

$$(\boldsymbol{I} - \boldsymbol{P}_H^{\mathrm{d}})\mathbf{curl}\,\boldsymbol{p} = \mathbf{curl}\,\boldsymbol{m} + \mathbf{grad}_h r, \quad r \in S_h, \quad \boldsymbol{m} \in \mathbf{curl}_h \boldsymbol{V}_h.$$

From the first estimate of Proposition 4.3 and the fact that $\boldsymbol{u}$ is divergence-free, we have that

$$\|\mathbf{grad}_h s\| \le cH\|\boldsymbol{u} - \boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{div})} \le cH\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{div})} = cH\|\boldsymbol{u}\|.$$

Again using the vanishing of $\operatorname{div}\boldsymbol{u}$, we obtain

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{p}\|^2 &= \Lambda^{\mathrm{d}}(\mathbf{curl}\,\boldsymbol{p},[\boldsymbol{I}-\boldsymbol{P}_H^{\mathrm{d}}]\boldsymbol{u}) = \Lambda^{\mathrm{d}}([\boldsymbol{I}-\boldsymbol{P}_H^{\mathrm{d}}]\mathbf{curl}\,\boldsymbol{p},\boldsymbol{u})\\ &= ([\boldsymbol{I}-\boldsymbol{P}_H^{\mathrm{d}}]\mathbf{curl}\,\boldsymbol{p},\boldsymbol{u}) = (\mathbf{curl}\,\boldsymbol{m},\boldsymbol{u}) = (\boldsymbol{m},\mathbf{curl}_h\boldsymbol{u}).\end{aligned}$$

From the second estimate of Proposition 4.3 we then get

$$\begin{aligned}\|\boldsymbol{m}\| &\le cH\|(\boldsymbol{I}-\boldsymbol{P}_H^{\mathrm{d}})\mathbf{curl}\,\boldsymbol{p}\|_{\boldsymbol{H}(\mathrm{div})}\\ &\le cH\|\mathbf{curl}\,\boldsymbol{p}\|_{\boldsymbol{H}(\mathrm{div})} = cH\|\mathbf{curl}\,\boldsymbol{p}\|.\end{aligned}$$

Hence, $\|\mathbf{curl}\,\boldsymbol{p}\| \le cH\|\mathbf{curl}_h\boldsymbol{u}\|$. Finally,

$$\|\boldsymbol{u}-\boldsymbol{P}_H^{\mathrm{d}}\boldsymbol{u}\| \le \|\mathbf{curl}\,\boldsymbol{p}\| + \|\mathbf{grad}_h s\| \le cH(\|\boldsymbol{u}\| + \|\mathbf{curl}_h\boldsymbol{u}\|),$$

which, together with 5.7, establishes 5.14. □

Proof of Proposition 4.4. Since

$$(\mathbf{grad}\,w_h,\mathbf{grad}\,\mu) = 0, \quad \mu \in W_H,$$

it follows from the standard duality argument, exploiting convexity, that

$$\|w_h\| \le cH\|\mathbf{grad}\,w_h\| \le cH\|\boldsymbol{p}-\boldsymbol{P}_H^c\boldsymbol{p}\|.$$

To prove the second estimate, we note that since $\boldsymbol{\Lambda}_h^c$ maps $\mathbf{curl}_h\boldsymbol{V}_h$ onto itself, we have $\boldsymbol{r}_h = (\boldsymbol{\Lambda}_h^c)^{-1}\mathbf{curl}_h\boldsymbol{v}_h \in \mathbf{curl}_h\boldsymbol{V}_h$. By Lemma 5.2 there exists $\boldsymbol{r}_H \in \boldsymbol{Q}_H$ such that

$$\|\boldsymbol{r}_h-\boldsymbol{r}_H\| \le cH\|\mathbf{curl}\,\boldsymbol{r}_h\|,\ \|\mathbf{curl}\,(\boldsymbol{r}_h-\boldsymbol{r}_H)\| \le cH\|\mathbf{curl}_h\mathbf{curl}\,\boldsymbol{r}_h\|.$$

Thus

$$\begin{aligned}\|\boldsymbol{r}_h-\boldsymbol{r}_H\|_{\Lambda_h^{\mathrm{c}}}^2 &\le cH^2(\|\mathbf{curl}\,\boldsymbol{r}_h\|^2 + \kappa^2\|\mathbf{curl}_h\mathbf{curl}\,\boldsymbol{r}_h\|^2)\\ &\le cH^2\kappa^{-2}(\|\boldsymbol{r}_h\|^2 + 2\kappa^2\|\mathbf{curl}\,\boldsymbol{r}_h\|^2 + \kappa^4\|\mathbf{curl}_h\mathbf{curl}\,\boldsymbol{r}_h\|^2)\\ &= cH^2\kappa^{-2}\|\boldsymbol{\Lambda}_h^{\mathrm{c}}\boldsymbol{r}_h\|^2 = cH^2\kappa^{-2}\|\mathbf{curl}_h\boldsymbol{v}_h\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{curl}_h\boldsymbol{v}_h\|^2 &= \Lambda^{\mathrm{c}}(\mathbf{curl}_h\boldsymbol{v}_h,\boldsymbol{r}_h) = \Lambda^{\mathrm{c}}(\boldsymbol{p}-\boldsymbol{P}_H^c\boldsymbol{p},\boldsymbol{r}_h)\\ &= \Lambda^{\mathrm{c}}(\boldsymbol{p}-\boldsymbol{P}_H^c\boldsymbol{p},\boldsymbol{r}_h-\boldsymbol{r}_H) \le \|\boldsymbol{p}-\boldsymbol{P}_H^c\boldsymbol{p}\|_{\Lambda_h^{\mathrm{c}}}\|\boldsymbol{r}_h-\boldsymbol{r}_H\|_{\Lambda_h^{\mathrm{c}}}\\ &\le cH\kappa^{-1}\|\boldsymbol{p}-\boldsymbol{P}_H^c\boldsymbol{p}\|_{\Lambda^{\mathrm{c}}}\|\mathbf{curl}_h\boldsymbol{v}_h\|,\end{aligned}$$

as desired. □

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