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Mixed finite elements for elasticity

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Summary. There have been many efforts, dating back four decades, to develop stable mixed finite elements for the stress-displacement formulation of the plane elasticity system. This requires the development of a compatible pair of finite element spaces, one to discretize the space of symmetric tensors in which the stress field is sought, and one to discretize the space of vector fields in which the displacement is sought. Although there are number of well-known mixed finite element pairs known for the analogous problem involving vector fields and scalar fields, the symmetry of the stress field is a substantial additional difficulty, and the elements presented here are the first ones using polynomial shape functions which are known to be stable. We present a family of such pairs of finite element spaces, one for each polynomial degree, beginning with degree two for the stress and degree one for the displacement, and show stability and optimal order approximation. We also analyze some obstructions to the construction of such finite element spaces, which account for the paucity of elements available.

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1. Introduction

Let σ and u denote the stress and displacement fields engendered by a body force f acting on a linearly elastic body which occupies a planar region Ω and which is clamped on $\partial\Omega$. Then σ takes values in the space $\mathbb{S} = \mathbb{R}^{2\times 2}_{\mathrm{sym}}$

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of symmetric tensors and u in $\mathbb{R}^2$, and the pair (σ, u) is characterized as the unique critical point of the Hellinger–Reissner functional

$$\mathcal{J}(\tau, v) = \int_\Omega \left(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v\right) dx. \tag{1.1}$$

Here the compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is bounded and symmetric positive definite uniformly for $x \in \Omega$, and the critical point is sought among all $\tau \in H(\operatorname{div}, \Omega, \mathbb{S})$, the space of square-integrable symmetric matrix fields with square-integrable divergence, and all $v \in L^2(\Omega, \mathbb{R}^2)$, the space of square-integrable vector fields. Equivalently, $(\sigma, u) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ is the unique solution to the following weak formulation of the elasticity system:

$$\begin{aligned} &\int_\Omega (A\sigma : \tau + \operatorname{div} \tau \cdot u + \operatorname{div} \sigma \cdot v)\, dx = \int_\Omega f v\, dx, \\ &(\tau, v) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2). \end{aligned} \tag{1.2}$$

A mixed finite element method determines an approximate stress field σ_h and an approximate displacement field u_h as the critical point of $\mathcal{J}$ over $\Sigma_h \times V_h$ where $\Sigma_h \subset H(\operatorname{div}, \Omega, \mathbb{S})$ and $V_h \subset L^2(\Omega, \mathbb{R}^2)$ are suitable piecewise polynomial subspaces. Equivalently, the pair $(\sigma_h, u_h) \in \Sigma_h \times V_h$ is determined by the weak formulation (1.2), with the test space restricted to $\Sigma_h \times V_h$. As is well known, the subspaces Σ_h and V_h cannot be chosen arbitrarily. To ensure that a unique critical point exist and that it provides a good approximation of the true solution, they must satisfy the stability conditions from the theory of mixed methods [6,7]:

(A1) There exists a positive constant c_1 such that $\|\tau\|_{H(\operatorname{div})} \le c_1 \|\tau\|_{L^2}$ whenever $\tau \in \Sigma_h$ satisfies $\int_\Omega \operatorname{div} \tau \cdot v\, dx = 0$ for all $v \in V_h$.
(A2) There exists a positive constant c_2 such that for all nonzero $v \in V_h$ there exists nonzero $\tau \in \Sigma_h$ with $\int_\Omega \operatorname{div} \tau \cdot v\, dx \ge c_2 \|\tau\|_{H(\operatorname{div})} \|v\|_{L^2}$.

Condition (A2) is one of the two stability conditions of [6], while (A1) implies the other, and essentially equivalent to it in practice. However we shall establish a second set of conditions, which imply (A1) and (A2) and also some other useful properties of the mixed method. These conditions are:

(A1′) $\operatorname{div} \Sigma_h \subset V_h$.
(A2′) There exists a linear operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma$ for all $\sigma \in H^1(\Omega, \mathbb{S})$, where $P_h : L^2(\Omega, \mathbb{R}^2) \to V_h$ denotes the L^2-projection.

It is clear that condition (A1$'$) implies (A1) with $c_1 = 1$. In order to see that the so-called commuting diagram property (A2$'$) implies (A2) we have to invoke the fact that for each $v \in L^2(\Omega, \mathbb{R}^2)$ there exists a $\tau \in H^1(\Omega, \mathbb{S})$ such that

$$\operatorname{div} \tau = v, \quad \|\tau\|_{H^1} \le c_0 \|v\|_{L^2},$$

where c_0 is independent of v (this is discussed briefly in the following section). For $v \in V_h$ and with this choice of τ we have $\Pi_h \tau \in \Sigma_h$, $\operatorname{div} \Pi_h \tau = P_h \operatorname{div} \tau = P_h v = v$, and

$$\int_\Omega \operatorname{div} \Pi_h \tau \cdot v \, dx = \|v\|_{L^2}^2 \ge c_0^{-1} \|\tau\|_{H^1} \|v\|_{L^2} \ge c_2 \|\Pi_h \tau\|_{H(\operatorname{div})} \|v\|_{L^2},$$

where $c_2 = c_0^{-1}(1 + \|\Pi_h\|^2_{\mathcal{L}(H^1, L^2)})^{-1/2}$.

Despite four decades of effort, very few choices of finite element spaces have been constructed which satisfy conditions (A1) and (A2). In fact, the only ones known use composite elements, in which V_h consists of piecewise polynomials with respect to one triangulation of the domain, while Σ_h consists of piecewise polynomials with respect to a different, more refined, triangulation [4,12,13,19]. Because of the lack of suitable mixed elasticity elements, several authors have resorted to the use of Lagrangian functionals which are modifications of the Hellinger–Reissner functional given above [1,3,5,15–18], in which the symmetry of the stress tensor is enforced only weakly or abandoned altogether. Another modification is analyzed in [14], where the solution space is altered such that Σ_h is only required to be a subspace of $L^2(\Omega, \mathbb{S})$.

In this paper we present a family of mixed finite element spaces for the unmodified Hellinger–Reissner formulation. The spaces consist of piecewise polynomials with respect to a single arbitrary triangular subdivision of Ω and satisfy conditions (A1$'$) and (A2$'$) with constants c_1 and c_2 independent of the triangulation (assuming, as usual, uniform shape regularity of the triangulations). The space V_h we use to approximate the displacement simply consists of all piecewise polynomials of degree k with no interelement continuity constraints. The degree k may be any positive integer (with different values of k determining different finite element spaces on the same triangulation). The space Σ_h is, of course, more complicated. Restricted to a single simplex the elements of Σ_h consist of all piecewise polynomial matrix fields of degree at most $k+1$ together with the divergence-free matrix fields of degree $k+2$. Degrees of freedom are specified on each element which determine the interelement continuity and ensure that $\Sigma_h \subset H(\operatorname{div}, \Omega, \mathbb{S})$.

In order to present the main ideas as clearly as possible, we shall first analyze the lowest order family of elements. After some preliminaries in Sect. 2, in Sect. 3 we construct the spaces Σ_h and V_h in the case $k = 1$, by describing their restrictions to each triangle and giving a unisolvent set of local degrees of freedom. We establish a relation between these space and the Hermite quintic C^1 finite element space, and explain in general the relationship between stable finite elements for the Hellinger–Reissner principle and C^1 finite elements. This provides a major obstruction to the construction of the former. Our element involves vertex degrees of freedom, a fact which, as we shall show, is unavoidable. Because of this, the interpolation operator into Σ_h associated with the degrees of freedom is not bounded on $H^1(\Omega, \mathbb{S})$, and so, even though it satisfies the commutativity property of condition (A2′), it does not establish that condition. In the following section we modify the interpolation operator, maintaining the commutativity, and establish its boundedness and approximation properties. Then in Sect. 5 we complete the error analysis for the $k = 1$ case. In Sect. 6, we briefly describe the case $k \geq 1$, which is altogether analogous to the case $k = 1$. In a few brief final paragraphs we describe a variant of the lowest order element with fewer degrees of freedom, and mention some simple extensions.

2. Notation and preliminaries

We denote by $H^k(T, X)$ the Sobolev space consisting of functions with domain $T \subset \mathbb{R}^2$, taking values in the finite-dimensional vector space X, and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either $\mathbb{S}$, $\mathbb{R}^2$, or $\mathbb{R}$. In the latter case we may write simply $H^k(T)$. We will generally write $\|\cdot\|_k$ or $\|\cdot\|_{H^k}$ instead of $\|\cdot\|_{H^k(T,X)}$. We similarly denote by $\mathcal{P}_k(T, X)$ the space of polynomials on T with degree at most k.

If τ is a symmetric matrix field then its divergence, $\operatorname{div}\tau$, is the vector field obtained by applying the ordinary divergence operator to each row. The symmetric part of the gradient of a vector field v, denoted $\epsilon\, v$, is given by $\epsilon\, v = [\operatorname{grad} v + (\operatorname{grad} v)^{\mathrm{T}}]/2$.

Throughout the paper we assume that the elastic domain Ω is a simply connected polygonal domain in $\mathbb{R}^2$. Any smooth vector field on Ω may be realized as the divergence of a smooth symmetric matrix field. E.g., we may extend the vector field smoothly to a larger smoothly bounded domain, and then solve the equations of elasticity there with the extended vector field as body forces. The same argument shows that any vector field in $L^2(\Omega, \mathbb{R}^2)$ may be realized as the divergence of a matrix field in $H^1(\Omega, \mathbb{S})$, a fact we already invoked in the introduction and will use as well in the sequel. Further, a symmetric matrix field τ on a simply connected domain is divergence-free if and only if τ admits a potential, or Airy stress function,

$$\tau = Jq := \begin{pmatrix} \partial^2 q/\partial y^2 & -\partial^2 q/\partial x\partial y \\ -\partial^2 q/\partial x\partial y & \partial^2 q/\partial x^2 \end{pmatrix}.$$

The potential q is determined by τ up to addition of a linear polynomial. These considerations are summarized by the statement that the following sequence is exact (i.e., that the range of each map is the kernel of the following one):

$$\begin{aligned} 0 \longrightarrow \quad \mathcal{P}_1(\Omega) \quad &\xrightarrow{\subset} \quad C^\infty(\Omega) \\ \xrightarrow{J} C^\infty(\Omega,\mathbb{S}) \ &\xrightarrow{\operatorname{div}} C^\infty(\Omega,\mathbb{R}^2) \longrightarrow 0. \end{aligned}$$

This exact sequence is related, although rather indirectly, to the de Rham sequence for the domain the Ω [11]. We have stated it in terms of infinitely differentiable functions, but analogous results hold with less smoothness. E.g., the sequence

$$\begin{aligned} 0 \longrightarrow \quad \mathcal{P}_1(\Omega) \quad &\xrightarrow{\subset} H^2(\Omega) \xrightarrow{J} H(\operatorname{div},\Omega,\mathbb{S}) \\ \xrightarrow{\operatorname{div}} L^2(\Omega,\mathbb{R}^2) &\longrightarrow \quad 0 \end{aligned} \tag{2.1}$$

is also exact. There is a polynomial analogue of the sequence as well: for any integer $k \geq 0$ the sequence

$$\begin{aligned} 0 \longrightarrow \quad \mathcal{P}_1(\Omega) \quad &\xrightarrow{\subset} \mathcal{P}_{k+3}(\Omega) \xrightarrow{J} \mathcal{P}_{k+1}(\Omega,\mathbb{S}) \\ \xrightarrow{\operatorname{div}} \mathcal{P}_k(\Omega,\mathbb{R}^2) &\longrightarrow \quad 0 \end{aligned}$$

is exact. (To verify the surjectivity of the final divergence, it suffices to count dimensions and use the exactness of the sequence at the other points.) As we shall see a key ingredient in the development and analysis of mixed methods for elasticity is a discrete analogue of the exact sequence (2.1).

3. The finite element method in the lowest order case

In this section we present a mixed finite element method based on piecewise linear displacements and piecewise quadratic stresses, the latter augmented by some cubic shape functions. First we describe the finite elements on a single triangle $T \subset \Omega$. Define

$$\begin{aligned} \Sigma_T &= \mathcal{P}_2(T,\mathbb{S}) + \{\, \tau \in \mathcal{P}_3(T,\mathbb{S}) \mid \operatorname{div}\tau = 0 \,\} \\ &= \{\, \tau \in \mathcal{P}_3(T,\mathbb{S}) \mid \operatorname{div}\tau \in \mathcal{P}_1(T,\mathbb{R}^2)\}, \\ V_T &= \mathcal{P}_1(T,\mathbb{R}^2). \end{aligned} \tag{3.1}$$

The space V_T has dimension 6 and a complete set of degrees of freedom are given by the value of the two components at the three nodes interior to T. The space Σ_T clearly has dimension at least 24, since the $\dim \mathcal{P}_3(T,\mathbb{S}) = 30$ and the condition that $\operatorname{div}\tau \in \mathcal{P}_1(T,\mathbb{R}^2)$ represents six linear constraints.

We now exhibit 24 degrees of freedom $\Sigma_T \to \mathbb{R}$ and show that they vanish simultaneously only when $\tau = 0$. This implies that the dimension of Σ_T is precisely 24 (which could also be established directly using the fact that $\operatorname{div} \mathcal{P}_3(\Omega, \mathbb{S}) = \mathcal{P}_2(\Omega, \mathbb{R}^2)$), and that the degrees of freedom are unisolvent. The degrees of freedom are

– the values of three components of $\tau(x)$ at each vertex x of T (9 degrees of freedom)
– the values of the moments of degree 0 and 1 of the two normal components of τ on each edge e of T (12 degrees of freedom)
– the value of the three components of the moment of degree 0 of τ on T (3 degrees of freedom)

Otherwise stated, we determine $\tau \in \Sigma_T$ by giving its values at the vertices, the values of $\int_e (\tau n)\, ds$ and $\int_e (\tau n) s\, ds$ for all edges, and the value of $\int_T \tau\, dx$. (Here s is a parameter giving the distance to one of the end points of e and n is one of the unit vectors normal to e.) Note that the degrees of freedom associated to an edge and its end points determine τn on that edge. This is just the condition required to obtain a conforming approximation of $H(\operatorname{div}, \Omega, \mathbb{S})$. The element diagrams in Fig. 1 are mnemonic of the degrees of freedom.

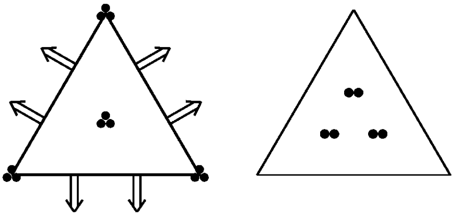

Fig. 1. Element diagrams for the lowest order stress and displacement elements

Lemma 3.1. *If the* 24 *degrees of freedom just given all vanish for some* $\tau \in \Sigma_T$, *then* $\tau = 0$.

Proof. We immediately have that τn vanishes on each edge. Letting $v = \operatorname{div} \tau$, a linear vector field on T, we get

$$\int_T v^2\, dx = -\int_T \tau : \epsilon\, v\, dx + \int_{\partial T} \tau n \cdot v\, ds = 0 \tag{3.2}$$

since the integral of τ vanishes as well as τn. Thus τ is divergence-free and hence $\tau = Jq$ for some $q \in \mathcal{P}_5(T)$. Adjusting by a linear function we may take q to vanish at the vertices. Now $\partial^2 q/\partial s^2 = \tau n \cdot n = 0$ on each edge, whence q is identically zero on ∂T. This implies that the gradient of

q vanishes at the vertices. Since $\partial^2 q/\partial s\partial n = -\tau n \cdot t = 0$ on each edge (with t a unit vector tangent to the edge), we conclude that $\partial q/\partial n$ vanishes identically on ∂T as well. Since q has degree at most 5, it must vanish identically. □

Having given a unisolvent set of degrees of freedom for V_T and Σ_T, our finite element space is assembled in the usual way. Let $\mathcal{T}_h$ be some triangulation of Ω, i.e., a set of closed triangles with union $\bar{\Omega}$ and such that any two distinct non-disjoint elements of $\mathcal{T}_h$ meet in a common edge or vertex. The associated finite element element space V_h is then the space of all piecewise linear vector fields with respect to this triangulation, not subject to any interelement continuity conditions. The space Σ_h is the space of all matrix fields which belong piecewise to Σ_T, subject to the continuity conditions that the normal components are continuous across mesh edges and all components are continuous at mesh vertices. The first condition is necessary to ensure that $\Sigma_h \subset H(\mathrm{div}, \Omega, \mathbb{S})$, but the second condition is a further restriction not implied by the inclusion in $H(\mathrm{div}, \Omega, \mathbb{S})$. (This is analogous to the condition of continuity of second derivatives at mesh vertices for the Hermite quintic C^1 finite element. We shall see below that these two phenomena are in fact related, and we shall argue that the restriction to vertex continuity is unavoidable.) Note that the stability condition (A1′) holds by construction.

The global degrees of freedom for the assembled finite element space Σ_h are the values of all three components at all the mesh vertices, the values of the moments of degrees 0 and 1 of the normal components on all the mesh edges, and the values of the moments of degree 0 for all components on all the mesh triangles. These functionals extend naturally to $C(\Omega, \mathbb{S})$ and so determine a canonical interpolation operator $C(\Omega, \mathbb{S}) \to \Sigma_T$. However, because of the vertex degrees of freedom, the canonical interpolation operator is not bounded with respect to the norm in $H^1(\Omega, \mathbb{S})$ and so cannot be used to establish the stability condition (A2′). Therefore we define an alternate interpolation operator which is bounded on $H^1(\Omega, \mathbb{S})$. To do so, we require bounded linear operators $E_h^x : H^1(\Omega, \mathbb{S}) \to \mathbb{S}$ for each vertex x of the triangulation. Given such operators, we define $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ by

$$\Pi_h \tau(x) = E_h^x \tau \quad \text{for all vertices } x, \tag{3.3}$$

$$\int_e (\tau - \Pi_h \tau) n \cdot v \, ds = 0 \quad \text{for all edges } e \text{ and all } v \in \mathcal{P}_1(e, \mathbb{R}^2), \tag{3.4}$$

$$\int_T (\tau - \Pi_h \tau) \, dx = 0 \quad \text{for all triangles } T. \tag{3.5}$$

If E_h^x were simply the evaluation operator $\tau \mapsto \tau(x)$, we would obtain the canonical interpolation operator, but this choice of E_h^x does not fulfill the required boundedness on $H^1(\Omega,\mathbb{S})$. A choice that is certainly bounded is simply $E_h^x\tau = 0$ for all τ and all x, and in fact this choice is sufficient for verifying the stability condition (A2′). We shall denote the resulting interpolation operator by Π_h^0. However, in the next section we shall choose $E_h^x\tau$ to be an approximate evaluation operator (a weighted average of τ near x), and in that way obtain an interpolation operator Π_h with better approximation properties.

We have for any $\tau \in H^1(\Omega,\mathbb{S})$, and $T \in \mathcal{T}_h$, and any $v \in V_T$ that

$$\int_T \operatorname{div}(\tau - \Pi_h\tau)\cdot v\,dx \\ = -\int_T (\tau - \Pi_h\tau) : \epsilon\, v\,dx + \int_{\partial T}(\tau - \Pi_h\tau)n\cdot v\,ds.$$

The right hand side vanishes in view of (3.5) and (3.4). This verifies the commutativity property

$$\operatorname{div}\Pi_h\tau = P_h \operatorname{div}\tau, \tag{3.6}$$

where $P_h : L^2(\Omega,\mathbb{R}^2) \to V_h$ is the orthogonal projection. We note that this property holds no matter how the choice of the E_h^x are made. A useful consequence of (3.6) is that $\operatorname{div}\Sigma_h = V_h$. Indeed, given any $v \in V_h$ we may find $\tau \in H^1(\Omega,\mathbb{S})$ such that $\operatorname{div}\tau = v$, and then $\Pi_h\tau \in \Sigma_h$ satisfies $\operatorname{div}\Pi_h\tau = P_h v = v$.

Let $Q_h = \{\, q \in H^2(\Omega) \mid Jq \in \Sigma_h \,\}$. It is easy to identify Q_h. Its elements are piecewise quintic polynomials and belong globally to $C^1(\Omega)$. Moreover, they are C^2 at the vertices of the mesh, since $Jq \in \Sigma_h$. Further, any piecewise quintic with these continuity properties is mapped by J into Σ_h, so belongs to Q_h. We conclude that Q_h is precisely the space of C^1 piecewise quintics which are C^2 at the vertices, that is, the well-known Hermite quintic or Argyris finite element [2]; cf., also [8, § 9]. The relationship between Q_h and Σ_h is even more intimate. Define a projection operator $I_h : C^\infty(\Omega) \to Q_h$ by requiring that the vertex values of $I_h q$, the vertex values of $\operatorname{grad} I_h q$, and the edge moments of degree 0 of $\partial(I_h q)/\partial n$ all be equal to the corresponding values for q, and that the Hessian of $I_h q$ at each vertex h be given by $JI_h q = E_h^x Jq$. Then the commutativity property $JI_h q = \Pi_h Jq$ can be verified easily. All these considerations are summarized in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & C^\infty(\Omega) & \xrightarrow{J} & C^\infty(\Omega,\mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(\Omega,\mathbb{R}^2) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle I_h} & & \big\downarrow{\scriptstyle \Pi_h} & & \big\downarrow{\scriptstyle P_h} & & \\
0 & \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & Q_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\operatorname{div}} & V_h & \longrightarrow & 0
\end{array}$$

It is instructive to examine which properties of our $H(\mathrm{div}, \Omega, \mathbb{S})$ finite element space Σ_h led us to an H^2 finite element space. Suppose we have any finite element space $\Sigma_h \subset H(\mathrm{div}, \Omega, \mathbb{S})$ and an interpolation operator $\Pi_h : C^\infty(\Omega, \mathbb{S}) \to \Sigma_h$ (e.g., the interpolation operation determined by a set of degrees of freedom for Σ_h) satisfing:

(a) $\Pi_h \tau$ is divergence-free whenever τ is divergence-free,
(b) $(\Pi_h \tau)n$ on any edge is determined by the restriction of τn to that edge,
(c) $\int_e (\Pi_h \tau)n\, ds = \int_e \tau n\, ds$ and $\int_e (\Pi_h \tau)n \cdot n\, s\, ds = \int_e \tau n \cdot n\, s\, ds$.

The first property is satisfied if a commutativity property like (3.6) holds, and the second is necessary if the interpolation operator maps into the space $H(\mathrm{div}, \Omega, \mathbb{S})$ and is determined by local degrees of freedom. The third property is usually required to verify the commutativity property (3.6). Under these assumptions we can define a corresponding subspace Q_h of $H^2(\Omega)$ as the inverse image of Σ_h under J (so, in particular, the elements of Q_h are piecewise polynomials of degree two greater than those of Σ_h), and define $I_h : C^\infty(\Omega) \to Q_h$ as follows. Given $q \in C^\infty(\Omega)$, $\Pi_h Jq$ is divergence-free by the second property, and so is equal to Jr for some function r determined up to addition of a linear polynomial. Thus for any specified vertex x we determine $I_h q$ uniquely by the conditions that $JI_h q = \Pi_h Jq$, $I_h q(x) = q(x)$, $\nabla I_h q(x) = \nabla q(x)$. Now $(Jq)n\cdot n = \partial^2 q/\partial s^2$ and $(Jq)n\cdot t = -\partial^2 q/\partial s \partial n$. Since $I_h q$ preserves the moments of Jq indicated in property (c), and it also preserves the value and the gradient of q at one vertex, we can integrate on edges to conclude that $I_h q = q$ and $\nabla I_h q = \nabla q$ at all vertices. We can then use property (b) in a similar way to show that $I_h q$ and $\partial I_h q/\partial n$ is determined on any edge by the the restriction of q and $\partial q/\partial n$ on that edge. This gives us the essential ingredients of a conforming H^2 finite element space determined by local degrees of freedom. Roughly speaking, we have shown that whenever one can construct an $H(\mathrm{div}, \Omega, \mathbb{S})$ finite element space Σ_h with expected properties, one can construct an H^2 finite element space related by $Q_h = J^{-1}(\Sigma_h)$. Our element is so related to the Hermite quintic element, the element of [13] is so related to the Clough–Tocher composite H^2 element, and the element family of [4] is so related to the higher order composite H^2 elements of [10].

The relationship between $H(\mathrm{div}, \Omega, \mathbb{S})$ finite elements and H^2 finite elements just outlined presents a major obstruction toward the construction of the former, since the difficulty in constructing conforming H^2 elements is well-known. There is an analogous obstruction to the development of $H(\mathrm{div}, \Omega, \mathbb{R}^2)$ finite elements, but it is, in contrast, very minor. It requires only that the inverse image of such a space under the curl operator must be a conforming finite element discretization of $H^1(\Omega)$, as, for example, the Lagrange finite elements of degree $k+1$ are the inverse image of the Raviart–Thomas elements of order k.

The finite element method for elasticity based on our spaces Σ_h and V_h has many features in common with popular mixed finite element methods for scalar elliptic problems, and we shall use these in Sect. 5 to give an error analysis. However there are some notable differences as well. One of these is the presence of vertex nodes. This leads to technical complications in the analysis, since it adds an additional regularity requirement for functions to belong to the domain of Π_h. However, as we now explain, vertex nodes are unavoidable whenever continuous shape functions are used to construct an $H(\mathrm{div}, \Omega, \mathbb{S})$ finite element space. To see why, imagine building an $H(\mathrm{div}, \Omega, \mathbb{S})$ finite element space from spaces Σ_T of continuous symmetric matrix fields, imposing interelement continuity only by means of quantities defined on the edges (and so shared by only two neighboring elements). This means that the degrees of freedom associated with each edge must determine the normal component on the edge. Now consider two edges of a triangle meeting at a common vertex x. If n_1 is the normal to the first edge and n_2 the normal to the second edge then the degrees of freedom on the first edge must determine τn_1 there and similarly the degrees of freedom on the second edge must determine τn_2 there. Since τ is continuous, we have in particular that the degrees of freedom on the first edge determine $\tau(x)n_1 \cdot n_2$, and those on the second edge determine $\tau(x)n_2 \cdot n_1$. But these quantities are equal since τ is symmetric. This is a contradiction, since the degrees of freedom on the two edges are necessarily independent.

This argument indicates that it is at least necessary to take the quantity $\tau(x)n_1 \cdot n_2$ as a degree of freedom associated to the vertex node x. But the node x is shared by other triangles which will have other values for the edge normal vectors n_i. For this reason, except if we restrict to very special triangulations, we are forced to take all three components of $\tau(x)$ as degrees of freedom associated to the node x[1]. Note that the composite $H(\mathrm{div}, \Omega, \mathbb{S})$ finite elements of [13] and [4] avoid the necessity of vertex degrees of freedom because they use discontinuous shape functions.

4. Approximation properties

In this section we analyze the approximation properties of the interpolation operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$. Recall that this operator is defined by (3.3)–(3.5), where E_h^x is a suitable approximate evaluation operator. We start by specifying our choice of E_h^x precisely. Let $\Sigma_h' = \Sigma_h \cap H^1(\Omega, \mathbb{S})$ be the space of continuous piecewise quadratic symmetric matrix fields, and let $R_h : L^2(\Omega, \mathbb{S}) \to \Sigma_h' \subset \Sigma_h$ be a Clement interpolant [9] satisfying

[1] A variant of this argument can be used to show that an H^2 finite element which uses C^1 shape functions must include second derivatives at the vertices among its degrees of freedom

(4.1) $$\|R_h\tau - \tau\|_j \le ch^{m-j}\|\tau\|_m, \quad 0 \le j \le 1, \quad j \le m \le 3,$$

where h denotes the mesh size (i.e., the maximum diameter of a triangle in $\mathcal{T}_h$), and the constant c depends only on the shape regularity of the triangulation. We then specify the interpolation operator Π_h via (3.3)–(3.5) with $E_h^x\tau = R_h\tau(x)$.

Recall also that $\Pi_h^0 : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ denotes the simpler interpolation operator obtained with the choice $E_h^x\tau = 0$. It is straightforward to verify the identity

$$\Pi_h = \Pi_h^0(I - R_h) + R_h.$$

Hence the error in Π_h, is given by

(4.2) $$I - \Pi_h = (I - \Pi_h^0)(I - R_h).$$

The operator Π_h^0 is completely local with respect to the triangulation $\mathcal{T}_h$ and we may analyze it triangle-by-triangle, mapping each triangle to a fixed reference element and applying standard scaling arguments. We can then establish error estimates for the interpolation operator Π_h by combining estimates for Π_h^0 with the estimates (4.1) for the Clement interpolant.

Let $F : \hat{T} \to T$ be an affine isomorphism of the form $F\hat{x} = B\hat{x} + b$. Given $\hat{\tau} : \hat{T} \to \mathbb{S}$, define $\tau : T \to \mathbb{S}$ by the matrix Piola transform

(4.3) $$\tau(x) := B\hat{\tau}(\hat{x})B^{\mathrm{T}},$$

where $x = F\hat{x}$. Clearly this sets up a one-to-one correspondence between $L^2(T, \mathbb{S})$ and $L^2(\hat{T}, \mathbb{S})$. A direct computation also shows that

(4.4) $$\operatorname{div}\tau(x) = B \operatorname{div}\hat{\tau}(\hat{x}).$$

It follows that $\tau \in H(\operatorname{div}, T, \mathbb{S})$ if and only if $\hat{\tau} \in H(\operatorname{div}, \hat{T}, \mathbb{S})$ and that $\tau \in \Sigma_T$ if and only if $\hat{\tau} \in \Sigma_{\hat{T}}$. Indeed, Σ_T is the space of τ polynomial of degree 3 such that $\operatorname{div}\tau$ is polynomial of degree 1, and, by (4.3) and (4.4), these properties clearly transform.

Let $\Pi_T^0 : H^1(T, \mathbb{S}) \to \Sigma_T$ be the restriction of Π_h^0 to a single triangle, i.e.,

- $\Pi_T^0\tau(x) = 0$ for all vertices x of T,
- $\int_e(\tau - \Pi_h^0\tau)n \cdot v\, ds = 0$ for all edges e of T, and $v \in \mathcal{P}_1(e, \mathbb{R}^2)$,
- $\int_T(\tau - \Pi_h^0\tau)\, dx = 0$.

We claim that

(4.5) $$\Pi_T^0\tau(x) = B\Pi_{\hat{T}}^0\hat{\tau}(\hat{x})B^{\mathrm{T}},$$

where τ and $\hat{\tau}$ are related by (4.3) and $x = F\hat{x}$. To verify (4.5) we first note that

$$B\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})B^{\mathrm{T}} = 0$$

for each vertex of T. Furthermore,

$$\begin{aligned}\int_T B\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})B^{\mathrm{T}}\,dx &= (\det B)B\int_{\hat{T}}\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})\,d\hat{x}B^{\mathrm{T}}\\ &= (\det B)B\int_{\hat{T}}\hat{\tau}(\hat{x})\,d\hat{x}B^{\mathrm{T}}\\ &= \int_T \tau(x)\,dx.\end{aligned}$$

Hence, it only remains to verify that $B\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})B^{\mathrm{T}}$ has the edge moments required of $\Pi^0_T\tau(x)$. Let e be an edge of T, $\hat{e}$ the corresponding edge of $\hat{T}$, and let $v \in \mathcal{P}_1(e,\mathbb{R}^2)$. Since $B^{\mathrm{T}}n_e$ is normal to $\hat{e}$ it follows that

$$\begin{aligned}\int_e [B\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})B^{\mathrm{T}}]n_e\cdot v(x)\,ds &= \frac{|e|}{|\hat{e}|}B\int_{\hat{e}}\Pi^0_{\hat{T}}\hat{\tau}(\hat{x})B^{\mathrm{T}}n_e\cdot\hat{v}(\hat{x})\,d\hat{s}\\ &= \frac{|e|}{|\hat{e}|}B\int_{\hat{e}}\hat{\tau}(\hat{x})B^{\mathrm{T}}n_e\cdot\hat{v}(\hat{x})\,d\hat{s}\\ &= \int_e \tau(x)n_e\cdot v(x)\,ds,\end{aligned}$$

where $\hat{v}(\hat{x}) = v(x)$. The equation (4.5) is therefore verified.

The operator $\Pi^0_{\hat{T}}$ is bounded from $H^1(\hat{T},\mathbb{S})$ to $L^2(\hat{T},\mathbb{S})$. Therefore, a standard scaling argument using (4.5) gives

$$\|\Pi^0_h\tau\|_0 \le c(\|\tau\|_0 + h\|\tau\|_1) \tag{4.6}$$

where the constant c depends only on the shape regularity of the triangulation. From (4.1) and (4.6) we obtain

$$\|\Pi^0_h(I-R_h)\tau\|_0 \le c(\|(I-R_h)\tau\|_0 + h\|(I-R_h)\tau\|_1 \le ch^m\|\tau\|_m$$

for $1 \le m \le 3$. Hence, it follows from (4.1) and (4.2) that

$$\|\Pi_h\tau - \tau\|_0 \le ch^m\|\tau\|_m, \quad 1 \le m \le 3. \tag{4.7}$$

It follows in particular that

$$\|\Pi_h\tau\|_0 \le c\|\tau\|_1, \tag{4.8}$$

and so the stability property (A2′) holds. We also note that for the orthogonal projection operator P_h, we have the obvious error estimates

$$\|v - P_hv\|_0 \le ch^m\|v\|_m, \quad 0 \le m \le 2. \tag{4.9}$$

5. Error analysis

Having established the stability properties (A1′) and (A2′) for the spaces Σ_h and V_h, we conclude that the Hellinger–Reissner functional has a unique critical point over $\Sigma_h \times V_h$ and obtain the quasioptimal estimate [6, 7]

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_{L^2} \le c \inf_{\substack{\tau \in \Sigma_h \\ v \in V_h}} (\|\sigma - \tau\|_{H(\mathrm{div})} + \|u - v\|_{L^2}).$$

The infimum on the right hand side is easily seen to be $O(h^2)$ for smooth solutions. However we shall now state and prove more precise estimates, with an analysis very analogous to that for second order elliptic problems.

Theorem 5.1. *Let (σ, u) denote the unique critical point of the Hellinger–Reissner functional over $H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ and let (σ_h, u_h) denote unique critical point over $\Sigma_h \times V_h$, where Σ_h and V_h are the spaces defined in Sect. 3. Then*

$$\|\sigma - \sigma_h\|_0 \le ch^m \|\sigma\|_m, \quad 1 \le m \le 3,$$
$$\|\mathrm{div}\, \sigma - \mathrm{div}\, \sigma_h\|_0 \le ch^m \|\mathrm{div}\, \sigma\|_m, \quad 0 \le m \le 2,$$
$$\|u - u_h\|_0 \le ch^m \|u\|_{m+1}, \quad 1 \le m \le 2.$$

Proof. Recall that the exact solution $(\sigma, u) \in H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ is the unique solution of (1.2), i.e.,

$$\int_\Omega (A\sigma : \tau + \mathrm{div}\, \tau \cdot u)\, dx = 0, \quad \tau \in H(\mathrm{div}, \Omega, \mathbb{S}),$$
$$\int_\Omega \mathrm{div}\, \sigma \cdot v\, dx = \int_\Omega f \cdot v\, dx, \quad v \in L^2(\Omega, \mathbb{R}^2).$$

The mixed method solution $(\sigma_h, u_h) \in \Sigma_h \times V_h$ satisfies the corresponding discrete system

$$\int_\Omega (A\sigma_h : \tau + \mathrm{div}\, \tau \cdot u_h)\, dx = 0, \quad \tau \in \Sigma_h,$$
$$\int_\Omega \mathrm{div}\, \sigma_h \cdot v\, dx = \int_\Omega f \cdot v\, dx, \quad v \in V_h.$$

Subtracting we obtain error equations, which, since $\mathrm{div}\, \Sigma_h \subset V_h$, we may write as

$$\int_\Omega [A(\sigma - \sigma_h) : \tau + \mathrm{div}\, \tau \cdot (P_h u - u_h)]\, dx = 0, \quad \tau \in \Sigma_h, \tag{5.1}$$

$$\int_\Omega \mathrm{div}(\sigma - \sigma_h) \cdot v\, dx = 0, \quad v \in V_h. \tag{5.2}$$

The second equation immediately implies that

$$\operatorname{div}\sigma_h = P_h \operatorname{div}\sigma = \operatorname{div}\Pi_h\sigma, \tag{5.3}$$

where the last equality comes from (3.6). Hence, from (4.9), we immediately obtain the desired error bound for $\operatorname{div}\sigma$:

$$\|\operatorname{div}\sigma - \operatorname{div}\sigma_h\|_0 = \|(I - P_h)\operatorname{div}\sigma\|_0 \le ch^m\|\operatorname{div}\sigma\|_m, \quad 0 \le m \le 2. \tag{5.4}$$

Taking $\tau = \Pi_h\sigma - \sigma_h$ in (5.1) and invoking (5.3) we obtain

$$\int_\Omega A(\sigma - \sigma_h) : (\Pi_h\sigma - \sigma_h)\,dx$$
$$= \int_\Omega \operatorname{div}(\Pi_h\sigma - \sigma_h)\cdot(P_h u - u_h)\,dx = 0,$$

from which it easily follows that

$$\|\sigma - \sigma_h\|_A \le \|\sigma - \Pi_h\sigma\|_A,$$

where $\|\tau\|_A^2 := \int A\tau : \tau\,dx$. Since the norm $\|\cdot\|_A$ is equivalent to the L^2–norm it follows from (4.7) that

$$\|\sigma - \sigma_h\|_0 \le ch^m\|\sigma\|_m, \quad 1 \le m \le 3. \tag{5.5}$$

In order to establish the error estimate for the displacement, we recall that

$$\|u - P_h u\|_0 \le ch^2\|u\|_2.$$

Therefore, the desired estimate will follow from the bound

$$\|P_h u - u_h\|_0 \le ch^m\|u\|_{m+1} \quad 1 \le m \le 2. \tag{5.6}$$

Let $\tau \in H^1(\Omega, \mathbb{S})$ satisfy $\operatorname{div}\tau = P_h u - u_h$ with $\|\tau\|_1 \le c\|P_h u - u_h\|_0$. Then, in light of the commutativity property, (3.6), and the bound (4.8), $\operatorname{div}\Pi_h\tau = P_h u - u_h$ and $\|\Pi_h\tau\|_0 \le c\|P_h u - u_h\|_0$. Hence,

$$\begin{aligned}\|P_h u - u_h\|_0^2 &= \int_\Omega \operatorname{div}\Pi_h\tau\cdot(P_h u - u_h)\,dx\\ &= -\int_\Omega A(\sigma - \sigma_h) : \Pi_h\tau\,dx\\ &\le c\|\sigma - \sigma_h\|_0\|P_h u - u_h\|_0.\end{aligned}$$

Thus

$$\|P_h u - u_h\|_0 \le c\|\sigma - \sigma_h\|_0 \le ch^m\|\sigma\|_m \le ch^m\|u\|_{m+1}, \quad 1 \le m \le 3,$$

which contains (5.6). □

6. Higher order elements

The foregoing considerations generalize in a straightforward manner to elements of higher order. Let k be a positive integer, the case $k = 1$ being that considered heretofore. Generalizing (3.1), we define the finite element spaces on a single triangle:

$$\begin{aligned}\Sigma_T &= \mathcal{P}_{k+1}(T,\mathbb{S}) + \{\, \tau \in \mathcal{P}_{k+2}(T,\mathbb{S}) \mid \operatorname{div}\tau = 0\,\} \\ &= \{\, \tau \in \mathcal{P}_{k+2}(T,\mathbb{S}) \mid \operatorname{div}\tau \in \mathcal{P}_k(T,\mathbb{R}^2)\}, \\ V_T &= \mathcal{P}_k(T,\mathbb{R}^2).\end{aligned}$$

Clearly $\dim V_T = (k+1)(k+2)$, and

$$\begin{aligned}\dim \Sigma_T \le d_k :&= \dim \mathcal{P}_{k+2}(T,\mathbb{S}) - [\dim \mathcal{P}_{k+1}(T,\mathbb{R}^2) - \dim \mathcal{P}_k(T,\mathbb{R}^2)] \\ &= (3k^2 + 17k + 28)/2.\end{aligned}$$

We shall show that in fact $\dim \Sigma_T = d_k$ and exhibit a unisolvent set of degrees of freedom. To this end, we define

$$M_k(T) = \{\, \tau \in \mathcal{P}_{k+2}(T,\mathbb{S}) \mid \operatorname{div}\tau = 0 \text{ and } \tau n = 0 \text{ on } \partial T\,\}.$$

In the proof of Lemma 3.1, we showed that for $k = 1$, $M_k(T) = 0$. The same argument shows that for $k \ge 2$, $M_k(T) = J(b_T^2 \mathcal{P}_{k-2}(T))$, where b_T is the cubic bubble function on T (the unique cubic polynomial achieving a maximum value of unity on T and vanishing on ∂T). Thus $\dim M_k(T) = (k^2 - k)/2$ for $k \ge 1$. The space $\epsilon[\mathcal{P}_k(T,\mathbb{R}^2)]$ has dimension $(k+1)(k+2) - 3$ and is clearly L^2-orthogonal to $M_k(T)$. Thus the space $N_k(T) := \epsilon[\mathcal{P}_k(T,\mathbb{R}^2)] + M_k(T)$ has dimension $(3k^2 + 5k - 2)/2$. The degrees of freedom we take for Σ_T are

- the values of three components of $\tau(x)$ at each vertex x of T (9 degrees of freedom)
- the values of the moments of degree at most k of the two normal components of τ on each edge e of T ($6k + 6$ degrees of freedom)
- the value of the moments $\int_T \tau : \phi \, dx$, $\phi \in N_k(T)$ ($(3k^2 + 5k - 2)/2$ degrees of freedom)

We have specified d_k degrees of freedom. Thus to show unisolvence it suffices to show that if all the degrees of freedom vanish for some $\tau \in \Sigma_k(T)$, then τ vanishes. The proof is just as for Lemma 3.1. From the vanishing of the first two sets of degrees of freedom we conclude that τn vanishes on ∂T. This, together with the vanishing of $\int_T \tau : \phi \, dx$, $\phi \in \epsilon[\mathcal{P}_k(T,\mathbb{R}^2)]$ implies that $\operatorname{div}\tau$ vanishes as well, so $\tau \in M_k(T)$, and, using the remaining degrees of freedom, we conclude that τ vanishes identically. Element

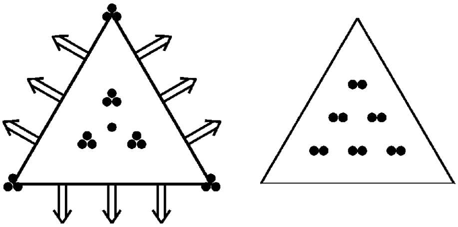

Fig. 2. Element diagrams for the stress and displacement elements in the case $k = 2$

diagrams for the case $k = 2$ are shown in Fig. 2. In this case $N_2(T)$ is the span of $\mathcal{P}_1(\mathcal{T}, \mathbb{S})$ and the additional matrix field $J(b_T^2)$.

Having specified finite element spaces on individual triangles and the degrees of freedom, we have determined the finite element spaces Σ_h and V_h for any triangulation. The latter space is simply the space of piecewise polynomial vector fields of degree at most k, while the former space consists of $H(\mathrm{div}, \Omega, \mathbb{S})$ matrix fields which belong to Σ_T on each $T \in \mathcal{T}_h$ and in addition are continuous at the mesh vertices. In addition to the natural interpolation operator determined by the degrees of freedom, we define $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ by using as degrees of freedom at the vertices the values of a suitable Clement interpolant. In this way we obtain an operator which satisfies the commutativity property (3.6) (with P_h the L^2-projection into V_h), and the error estimates (4.7), now for $1 \le m \le k+2$. The analysis generalizes directly from the case $k = 1$, resulting in the following analogue of Theorem 5.1.

Theorem 6.1. *Let (σ, u) denote the unique critical point of the Hellinger–Reissner functional over $H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ and let (σ_h, u_h) denote the unique critical point over $\Sigma_h \times V_h$, where Σ_h and V_h are the spaces defined above for some integer $k \ge 1$. Then*

$$\|\sigma - \sigma_h\|_0 \le ch^m \|\sigma\|_m, \quad 1 \le m \le k+2,$$
$$\|\mathrm{div}\, \sigma - \mathrm{div}\, \sigma_h\|_0 \le ch^m \|\mathrm{div}\, \sigma\|_m, \quad 0 \le m \le k+1,$$
$$\|u - u_h\|_0 \le ch^m \|u\|_{m+1}, \quad 1 \le m \le k+1.$$

7. A simplified element of low order

There does not exist an element pair for $k = 0$ in our family. Indeed, there could not, since there does not exist an H^2 finite element of degree 4. However, there is a variant of the lowest order ($k = 1$ element), with fewer degrees of freedom, which we now describe briefly. To do so we denote by $RM(T)$ the space of infinitesimal rigid motions on T, i.e., the span of $\mathcal{P}_0(T, \mathbb{R}^2)$ and the linear vector field $(-x_2, x_1)$. This is precisely the kernel

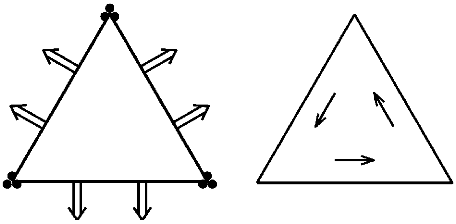

Fig. 3. Element diagrams for the elements of reduced order

of the symmetric gradient operator ϵ. For our simplified element we take V_T to be $RM(T)$ and

$$\Sigma_T = \{\, \tau \in \mathcal{P}_3(T, \mathbb{S}) \mid \operatorname{div} \tau \in RM(T) \,\}.$$

Then $\dim \Sigma_T = 21$ and a unisolvent set of degrees of freedom are

- the values of three components of $\tau(x)$ at each vertex x of T (9 degrees of freedom)
- the values of the moments of degree 0 and 1 of the two normal components of τ on each edge e of T (12 degrees of freedom)

See Fig. 3. In this case $\mathcal{P}_0(T, \mathbb{R}^2) \subsetneq V_T \subsetneq \mathcal{P}_1(T, \mathbb{R}^2)$ and $\mathcal{P}_1(T, \mathbb{S}) \subsetneq \Sigma_T \subsetneq \mathcal{P}_2(T, \mathbb{S})$. Therefore the final error estimates are

$$\begin{gathered} \|\sigma - \sigma_h\|_0 \le ch^m \|\sigma\|_m, \quad 1 \le m \le 2, \\ \|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_0 \le ch^m \|\operatorname{div} \sigma\|_m, \quad 0 \le m \le 1, \\ \|u - u_h\|_0 \le ch \|u\|_2. \end{gathered}$$

The analysis follows closely that of the previous element family. One difference is that the space Σ_T for this element is not invariant under the Piola transform. Consequently a different argument is required to prove the approximation properties of Σ_h. This can be done, for example, by scaling to a similar element of unit diameter using translation, rotation, and dilation, and using a compactness argument.

8. Final remarks

In the interest of clarity we have considered an elastic body clamped all around its boundary and described by a uniformly positive definite compliance tensor. Both restrictions can be loosened. The extension to traction boundary conditions on part of the boundary is straightforward. For the Hellinger–Reissner variational form, traction boundary conditions are essential, and thus must be imposed in the stress space Σ_h. When traction boundary conditions are imposed on both edges meeting at a corner, then the entire stress tensor must vanish there, and so all three degrees of freedom at the corner set equal to zero. At other boundary points traction boundary conditions imply two linear relations among three degrees of freedom. (Such linear relation boundary conditions can be implemented by modifying the relevant columns and rows of the unconstrained stiffness matrix, maintaining symmetry.)

Another generalization that can easily be handled is the extension to nearly incompressible or incompressible elastic materials. In the homogeneous isotropic case the compliance tensor is given by $A\tau = [\tau - \lambda/(2\mu + 2\lambda)\,\mathrm{tr}\,\tau\, I]/2\mu$, where $\mu > 0$, $\lambda \geq 0$ are the Lamé constants. For our mixed method, as for most methods based on the Hellinger–Reissner principle, one can prove that the error estimates hold uniformly in λ (the incompressible limit being $\lambda \to \infty$). In the analysis above we used the fact that $\int A\tau : \tau\, dx \geq c_0 \|\tau\|_0^2$ for some positive c_0. This estimate degenerates ($c_0 \to 0$) as $\lambda \to \infty$. However the estimate remains true with $c_0 > 0$ depending only on Ω and μ if we restrict τ to functions for which $\mathrm{div}\,\tau \equiv 0$, $\int_\Omega \mathrm{tr}\,\tau\, dx = 0$, and this is enough to carry through the analysis. See [4] where the details are presented for a composite mixed element.

References

1. Mohamed Amara, Jean-Marie Thomas: Equilibrium finite elements for the linear elastic problem. Numer. Math. **33**(4), 367–383 (1979)
2. John H. Argyris, Isaac Fried, Dieter W. Scharpf: The TUBA family of plate elements for the matrix displacement method. Aero. J. Roy. Aero. Soc. **72**, 514–517 (1968)
3. Douglas N. Arnold, Franco Brezzi, Jim Douglas, Jr.: PEERS: a new mixed finite element for plane elasticity. Japan J. Appl. Math. **1**(2), 347–367 (1984)
4. Douglas N. Arnold, Jim Douglas, Jr., Chaitan P. Gupta: A family of higher order mixed finite element methods for plane elasticity. Numer. Math. **45**(1), 1–22 (1984)
5. Douglas N. Arnold, Richard S. Falk: A new mixed formulation for elasticity. Numer. Math. **53**(1-2), 13–30 (1988)
6. Franco Brezzi: On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8**(R-2), 129–151 (1974)
7. Franco Brezzi, Michel Fortin: Mixed and Hybrid Finite Element Methods. New York: Springer-Verlag, 1991
8. Philippe G. Ciarlet: The finite element method for elliptic problems. Amsterdam: North-Holland, 1978
9. Philippe Clément: Approximation by finite element functions using local regularization. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **9**(R-2), 77–84 (1975)
10. Jim Douglas, Jr., Todd Dupont, Peter Percell, Ridgway Scott: A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems. RAIRO Anal. Numér. **13**(3), 227–255 (1979)
11. Michael Eastwood: A complex from linear elasticity. Rend. Circ. Mat. Palermo (2) Suppl. (2000), no. 63, 23–29

12. Baudoiun M. Fraejis de Veubeke: Displacement and equilibrium models in the finite element method. Stress analysis, pp. 145–197 (O.C Zienkiewics, G.S. Holister, eds.), New York, Wiley, 1965
13. Claes Johnson, Bertrand Mercier: Some equilibrium finite element methods for two-dimensional elasticity problems. Numer. Math. **30**(1), 103–116 (1978)
14. Alain L. Mignot, Claude Surry: A mixed finite element family in plane elasticity. Appl. Math. Modelling **5**, 259–262 (1981)
15. Erwin Stein, Raimund Rolfes: Mechanical conditions for stability and optimal convergence of mixed finite elements for linear plane elasticity. Comput. Methods Appl. Mech. Engrg. **84**(1), 77–95 (1990)
16. Rolf Stenberg: On the construction of optimal mixed finite element methods for the linear elasticity problem. Numer. Math. **48**(4), 447–462 (1986)
17. Rolf Stenberg: A family of mixed finite elements for the elasticity problem. Numer. Math. **53**(5), 513–538 (1988)
18. Rolf Stenberg: Two low-order mixed methods for the elasticity problem. The mathematics of finite elements and applications, VI (Uxbridge, 1987), Academic Press, London, 1988, pp. 271–280
19. Vernon B. Watwood Jr., B. J. Hartz: An equilibrium stress field model for finite element solution of two–dimensional elastostatic problems. Internat. Jour. Solids and Structures **4**, 857–873 (1968)