Möbius Transformations Revealed

Douglas N. Arnold and Jonathan Rogness

Möbius Transformations Revealed is a short film that illustrates a beautiful correspondence between Möbius transformations and motions of the sphere. The video received an Honorable Mention in the 2007 Science and Engineering Visualization Challenge, cosponsored by the National Science Foundation and *Science* magazine. It subsequently received extensive coverage from both traditional media outlets and online blogs. Edward Tufte, the world's leading expert on the visual display of information, came across the video and reported on his blog "*Möbius Transformations Revealed* is a wonderful video clarifying a deep topic… amazing work…" But the film has also attracted a far less expert audience. As of this writing, it has been viewed nearly 1.5 million times on the video-sharing website YouTube and is rated as the number three top favorite video of all time in YouTube's educational category. Over 11,000 viewers have declared it among their favorites, which makes it one of the YouTube top favorites of all time. From the more than 4,000 written comments left by YouTube viewers it is clear that many of them had little mathematical background, and some were quite young. To view *Möbius Transformations Revealed*, visit the website `http://umn.edu/~arnold/moebius/`.

In this article we discuss some of the technical details behind the video and offer a "behind the scenes" look at its production. We begin with a brief overview of the visualization of functions of a complex variable, especially the technique used throughout the video, which we refer to as *homotopic image mapping*. This is followed by a discussion of Möbius transformations and the specific theorem illustrated in the video. We conclude by describing aspects of the movie that are generally unnoticed by the public but can be appreciated by mathematicians.

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Visualization of Functions

Among the most insightful tools that mathematics has developed is the representation of a function of a real variable by its graph. In fact, historically, graphs of functions appeared before the notion of function itself. A graph of the inclinations of planets as a function of time appears already in a tenth century manuscript [1], and in the fourteenth century Nicolas Oresme published a graphical method for displaying data that leads to graphs that appear quite familiar (see Figure 1).

By the late seventeenth and early eighteenth century, when the notion of function was developed by Leibniz, John Bernoulli, Euler, and others, graphs appeared in their works that would not be out of place in today's calculus texts. Who today would attempt to teach the trigonometric functions, without drawing a graph?

The situation is quite different for a function of a complex variable. The graph is then a surface in four-dimensional space, and not so easily drawn. Many texts in complex analysis are without a single depiction of a function. Nor is it unusual for average students to complete a course in the subject with little idea of what even simple functions, say trigonometric functions, "look like". (Fortunately

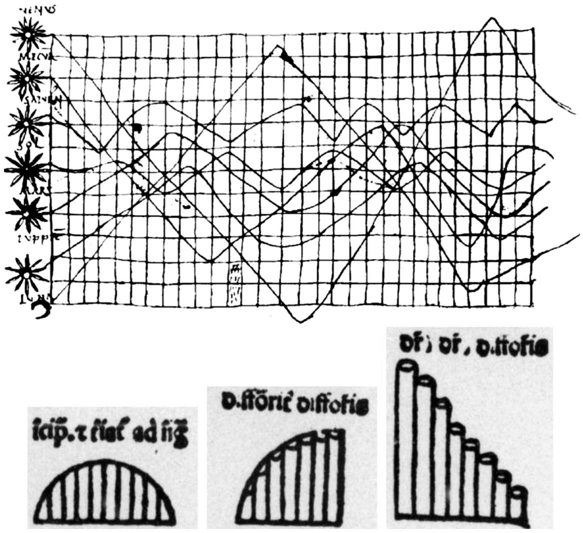

Figure 1. Tenth and fourteenth century graphs.

there are some exceptional textbooks, such as the lovely *Visual Complex Analysis* by Needham [2].)

The most straightforward way to visually represent a function $w = f(z)$ of a complex variable is to depict the image in the w-plane of some identifiable point set in the domain. At the simplest level, this may consist of finitely many labeled points or curves, but more information is transmitted by depicting the image of a region labeled with an easily identifiable pattern, such as a checkerboard (see Figure 2). With the aid of computer graphics, one can easily incorporate colors or even images.

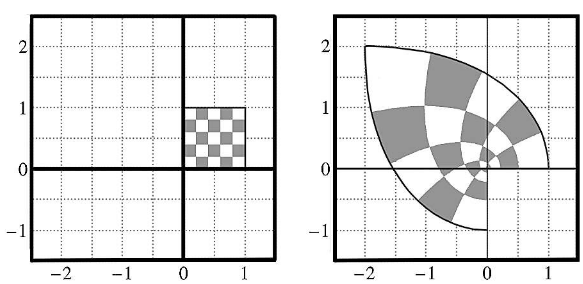

Figure 2. A checkerboard on the unit square and its image under $f(z) = z^3$.

This can help the viewer to ascertain the configuration of large parts of the image at a glance. See Figures 3 and 4. Note that this image mapping approach to visualizing complex functions offers a great deal of freedom in comparison to standard graphs of real functions, in which the only significant choices to be made are the ranges and scales of the axes. For complex image mapping we have the choice of the region in the z-plane to display and the *domain pattern*, i.e., the pattern, coloring, or other labeling of the region. Different choices

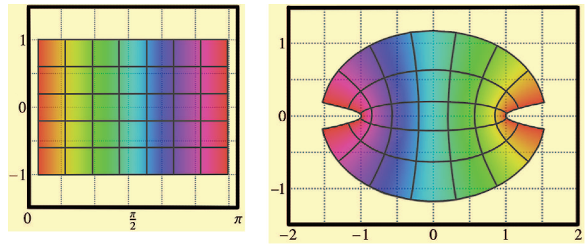

Figure 3. A colored rectangle and its image mapped via $\cos z$.

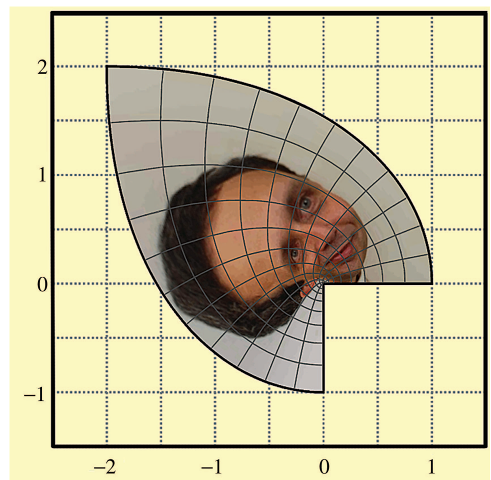

Figure 4. The first author's photo under z^3.

of domain pattern can significantly enhance the communication of salient features of the function. Transparency is a commonly implemented feature in many computer graphics systems, and the use of partially transparent domain patterns can help with the difficulties image mapping encounters with multivalent functions (see Figure 5). (An alternative method of depicting complex functions, called *domain coloring*, avoids the difficulties with multivalence by depicting the inverse image of a pattern in the range [3].)

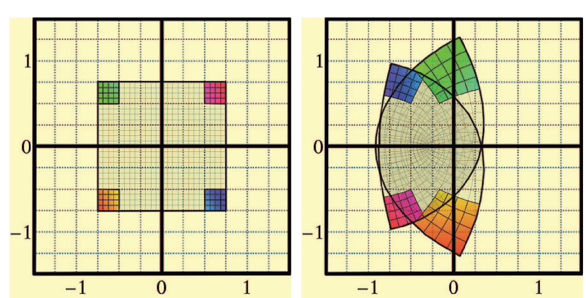

Figure 5. $f(z) = (2z^3 - 6z^2 - z - 2)/6$**.**

Even with a well-chosen domain pattern, it may be difficult to relate points in the image plane to their inverse images in the domain plane. Animation is a very effective tool in this regard. With *homotopic image mapping* we depict not just the image of the domain pattern under the mapping f, but the evolution of the image under a homotopy connecting the identity map to f. In this way a great deal of information can be conveyed quickly. A glance at even four frames from such a homotopy, shown in Figure 6, makes it easy to see how the complex exponential map takes the rectangle $|\operatorname{Re}(z)| \le a$, $|\operatorname{Im}(z)| \le \pi$ onto the annulus $e^{-a} \le |w| \le e^{a}$. In this example, we use a simple linear homotopy, $F(z,t) = (1-t)z + t\exp(z)$, $0 \le t \le 1$, but the choice of homotopy is another factor that can be used to advantage. For example, to visualize the function $f(z) = z^3$ we might want to use a homotopy through power maps: $F(z,t) = z^{1+t}$, $0 \le t \le 2$.

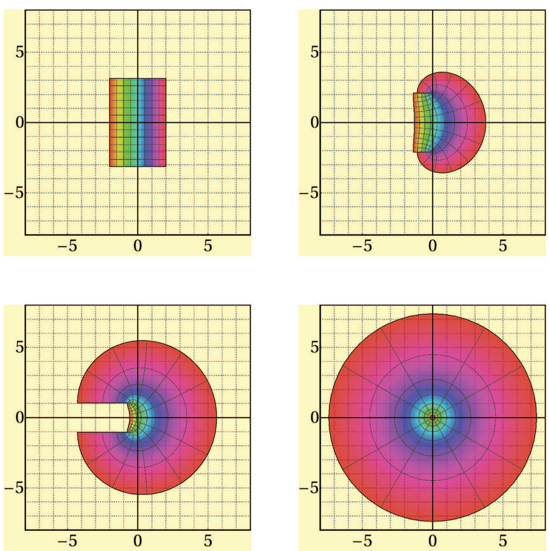

Figure 6. Homotopy to $f(z) = e^z$.

Möbius Transformations

Möbius transformations, i.e., non-constant rational maps of the form

$$f(z) = \frac{az+b}{cz+d},$$

are fundamental complex maps, useful in many applications, and studied in most courses on complex analysis. They are invertible meromorphic functions (in fact the group of meromorphic automorphisms of the extended complex plane $\mathbb{C}_\infty$ consists precisely of the Möbius transformations), and so are conformal everywhere. They also possess the less common geometrical property that they map arcs of circles (understood to include line segments as a limiting case) to arcs of circles. Thus Möbius transformations are natural candidates for visualization by image mapping. These can be animated effectively by using a homotopy consisting entirely of Möbius transformations that joins the identity (which is a Möbius transformation) to the given transformation. This technique is used extensively in *Möbius Transformations Revealed*.

The characterization of the Möbius transformations as the meromorphic automorphisms of the extended complex plane can be interpreted geometrically. The extended plane can be identified with the unit sphere in $\mathbb{R}^3$ as usual. Namely, we identify the complex plane with the plane $x_3 = 0$ in $\mathbb{R}^3$, and map it to the unit sphere by inverse stereographic projection from the north pole. Completing the identification by mapping the point at infinity in $\mathbb{C}_\infty$ to the north pole, the Möbius transformations correspond to the holomorphic automorphisms of this Riemann sphere. However, it is not obvious what the holomorphic automorphisms of the sphere look like, and it takes some effort and sophistication to get a clear picture of the Möbius transformations in this way.

Stereographic projection can be used to characterize Möbius transformations in a distinctly different way, which is both elegant and visually accessible. Call a sphere S in $\mathbb{R}^3$ admissible if its north pole lies in the upper half-space $H = \{x_3 > 0\}$, and, for such spheres, denote by P_S the stereographic projection from the north pole s_0 of S, which identifies $\mathbb{C}_\infty$ with S. Choose some such sphere, and also a rigid motion T of $\mathbb{R}^3$ such that $S' := TS$ is also admissible, i.e., $Ts_0 \in H$. Consider the composition $P_{S'} \circ T \circ P_S^{-1}$, which maps $\mathbb{C}_\infty$ to itself. It is easy to verify that the composition is a Möbius transformation, either by direct calculation, or, from a more advanced viewpoint, by noting that it corresponds to the map from S to itself given by $P_S^{-1} \circ P_{S'} \circ T$, which is surely a holomorphic automorphism.

In fact, every Möbius transformation is obtained in this way.

Theorem 1. *A complex mapping is a Möbius transformation if and only if it can be obtained by stereographic projection of the complex plane onto an admissible sphere in $\mathbb{R}^3$, followed by a rigid motion of the sphere in $\mathbb{R}^3$ which maps it to another admissible sphere, followed by stereographic projection back to the plane.*

We have not been able to ascertain the origin of this simple, elegant result. A broad, if unscientific, survey of colleagues indicates that the theorem is known by some, but no one has been able to provide a concrete reference. In 2006 it was added by an anonymous contributor as the second sentence of the article on Möbius transformations in the web-based free content encyclopedia Wikipedia ("A Möbius transformation may be performed by performing a stereographic projection from a plane to a sphere, rotating and moving that sphere to a new arbitrary location and orientation, and performing a stereographic projection back to the plane.") More recently, in 2008 this sentence was removed from Wikipedia by a contributor whose comments indicate a misunderstanding of the result.

To prove Theorem 1, we must show that for any Möbius transformation f there exists an admissible sphere S and a rigid motion T such that $S' = TS$ is admissible and that

$$f = P_{S'} \circ T \circ P_S^{-1}. \tag{1}$$

We rely on the elementary fact that the Möbius transformations are generated by the translations $z \mapsto z + \alpha$ ($\alpha \in \mathbb{C}$), the rotations $z \mapsto e^{i\theta}$ ($\theta \in \mathbb{R}$), the dilations $z \mapsto \rho z$ ($\rho > 0$), and the inversion $z \mapsto 1/z$. In fact, it is easy to write any Möbius transformation (except a linear polynomial, which is an easier case) as

$$f(z) = \frac{\rho e^{i\theta}}{z + \alpha} + \beta \tag{2}$$

for appropriate $\alpha, \beta \in \mathbb{C}$ and $\rho, \theta \in \mathbb{R}$. In other words, f is obtained as the composition

(1) translation by α
(2) inversion
(3) dilation by ρ
(4) rotation by θ
(5) translation by β

Now, the translation by α may be realized in the form (1) by choosing S to be any admissible sphere and T to be the same translation extended to $\mathbb{R}^3$. For each of the other maps, rotation, dilation, and inversion, we choose S to the be the unit sphere. To obtain a rotation, of course, we take T to be the same rotation extended to $\mathbb{R}^3$ (rotation about the x_3-axis). To obtain dilation by ρ, we take T to be translation of the sphere *upwards* a distance $\rho - 1$. And to obtain the inversion, we take T to be rotation *around the real axis* of the complex plane through an angle π. Therefore, we can write the general Möbius transformation (2) in the form (1) by choosing S to be a sphere of unit radius centered at the point $-\alpha$ of the complex plane, and construct T as the composition of translation by α, rotation by π around the real axis, rotation by θ around the axis orthogonal to the plane, translation upwards by $\rho - 1$, and translation by β.

Note that the choice of the sphere S and rigid motion T are far from unique. After all, they offer ten degrees of freedom, while the Möbius group is just six-dimensional. An example of non-uniqueness is shown in Figure 7, which displays two representations of the Möbius transformation $f(z) = [(-1+i)z - \sqrt{2}]/[(-1+i)z + \sqrt{2}]$ using spheres of unit radius. In the first, T is a rotation about the center of the initial sphere S, so the final sphere S' coincides with S. In the second representation, T involves translation as well as rotation.

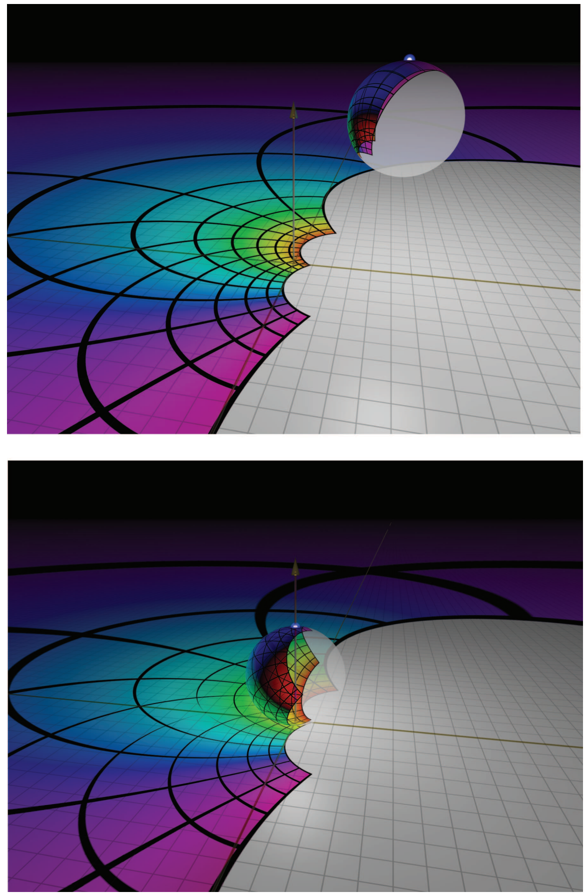

Figure 7. Distinct representations of the same Möbius transformation.

Möbius Transformations Revealed

The video *Möbius Transformations Revealed* demonstrates various geometric properties of Möbius transformations—e.g., conformality, circle-to-circle mapping, and generation by translations, rotations, dilations, and inversion—using homotopic image mapping. With the addition of 3-dimensional computer animation, it demonstrates the relation between Möbius transformations of the plane to stereographic projections of a sphere and gives a convincing demonstration of the elusive Theorem 1.

A very satisfying aspect of the production of the video is that the theorem it demonstrates—that Möbius transformations can be obtained by simple rigid motions of a sphere through 3-space via stereographic projection—was a key to producing the video itself. As we often teach in the classroom, stereographic projection is the mathematical realization of the physical process of illuminating a plane from a bright light placed at the far pole of a translucent sphere and following the light rays from the pole through the sphere onto the plane. See Figure 8. The frames of *Möbius Transformations Revealed* were produced using ray-tracing software. We used the Persistence of Vision Raytracer (POV-Ray), a totally free and widely available program that runs on most computer platforms. With ray-tracing, the user enters the configurations and attributes (such as texture, color, and transparency) of objects, light sources, and a camera in a virtual 3-dimensional world. The software then renders the 2-dimensional image seen by the camera as a result of light rays interacting with the objects.

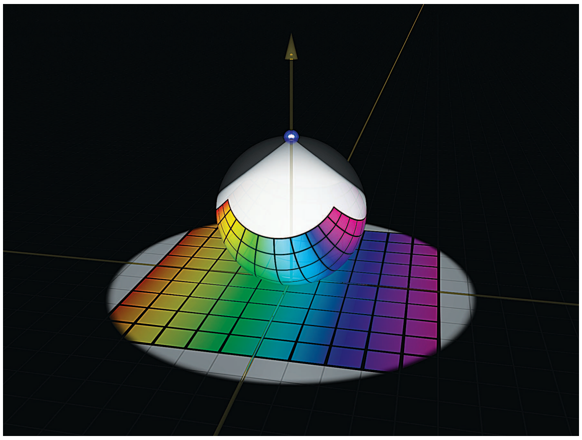

Figure 8. Stereographic projection.

The 3-dimensional world of *Möbius Transformations Revealed* is very simple. We needed only to provide a sphere, appropriately colored and translucent, a plane with appropriate markings (grid) and reflectivity, and a light source on the pole of the sphere opposite the plane. For each frame we positioned and oriented the sphere and the camera appropriately, and POV-Ray did the rest.

Of course this is oversimplified. The description for POV-Ray of a transparent sphere colored with a translucent image of a rainbow-colored square under inverse stereographic projection involves a messy calculation with spherical coordinates, and we used *Mathematica* to compute it. A fair amount of calculation was needed as well to choose the positions and orientations, and a lot of adjustment of visual attributes was needed to obtain images of high quality. In mathematical videos, as in other movies, production values are important, and thought has to be given to non-mathematical issues such as color choices, line thickness, viewing area, choice and depiction of axes, speed of the homotopies, etc. As with any educational activity, decisions had to be made about what to include and what to omit, and the level of presentation. *Möbius Transformations Revealed* could even be said to have a simple plot, in which the Möbius transformations are introduced in two dimensions, and then "revealed" by moving the camera from straight overhead, looking down at the plane, with the sphere invisible, to a side view in which the sphere becomes visible and can be seen together with the plane. It is set to music, and the affinity between the images on the screen and the selection from Schumann's *Kinderscenen* performed by pianist Donald Betts undoubtedly contributed to the popularity of the video.

An interesting aspect of the ray-traced frames in the 3-dimensional portion of the video, is that they combine the effects of stereographic projection onto a plane and perspective projection of the plane onto the camera's imaging plane. Because of the perspective projection, the image of a line segment under a Möbius transformation, i.e., the image under stereographic projection on the complex plane of a circular arc on the sphere, does not appear as a circular arc on the screen, but rather as an ellipse. In some cases, the eccentricity of the ellipse is large: circles with a large radius may appear to be nearly straight lines until they bend sharply in the distance; see Figure 9. As was discovered by artists during the Renaissance, a circle rendered as the appropriate ellipse via projection conveys a more genuine sense of a circle, than if it were rendered as a circle.

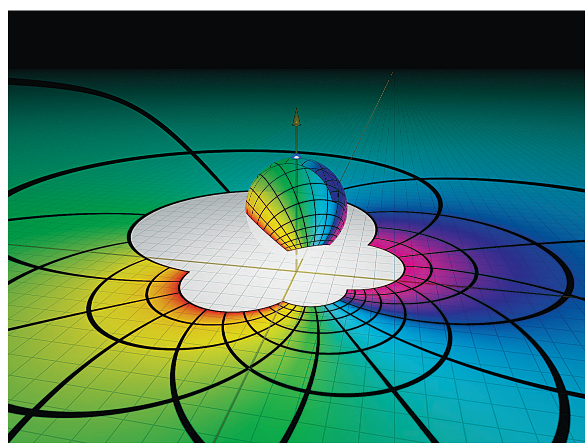

Figure 9. A circle appears to bend sharply at upper left.

In fact, the situation is more complicated. The black curves drawn as a grid on the sphere are not 1-dimensional curves at all, but have width. Therefore, even without the perspective projection, they would not be projected onto true circles and lines on the plane, but rather onto two-dimensional tubular neighborhoods of circles and lines. These neighborhoods become quite distorted when the curves are close to the light source. Figure 10, for example, shows a line which should be projected onto the positive imaginary axis, but in fact becomes arbitrarily wide. Readers will also notice the varying width of the circular arcs in the image. Again this contributes to a sense of reality of the image.

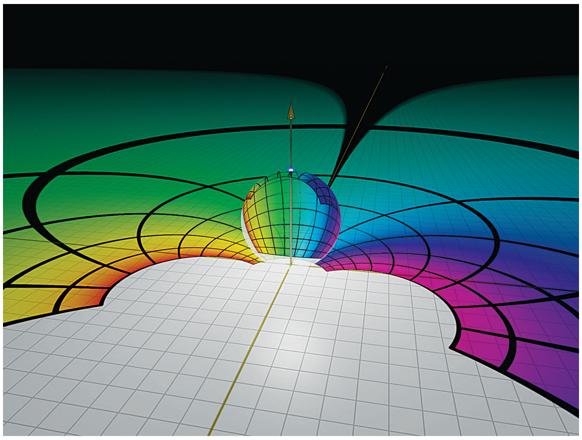

Figure 10. A thin segment becomes very thick near infinity.

The correspondence between stereographic projection and its ray-tracing realization illustrated in Figure 8 is not perfect. Mathematically, the case where the sphere intersects the plane is perfectly allowable, but the physical model of stereographic projection we used in the video breaks down in that case. Figure 11 shows this situation clearly. Inside the unit circle $|z| = 1$ light rays hit the plane before reaching the color on the sphere. We avoided choosing such spheres in *Möbius Transformations Revealed*.

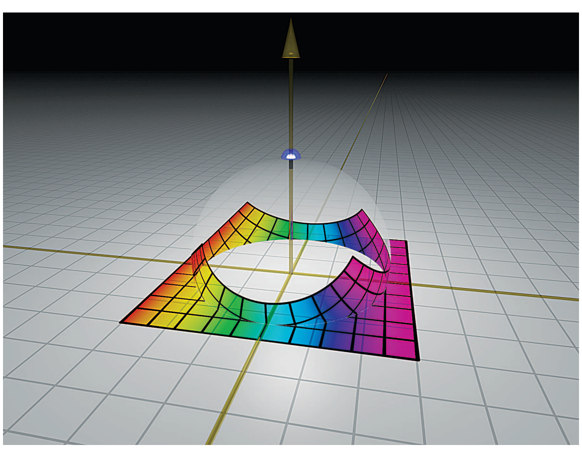

Figure 11. The physical model of stereographic projection fails when the sphere intersects the plane.

The first part of *Möbius Transformations Revealed* is 2-dimensional, but we still used ray-tracing to generate the Möbius transformations.

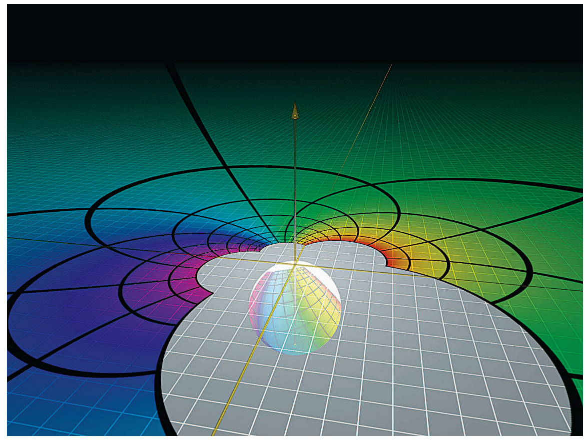

Figure 12. In the video, unlike in this image, the sphere underneath the plane is hidden from view and the camera points straight down to get a 2-dimensional view.

How did we hide the sphere? We placed the camera directly above the origin of the plane, looking down at it, but placed the sphere *underneath* the plane with the light source at the *south* pole, causing the plane to be illuminated by colors from below. See Figure 12, where the camera has been moved away from the z-axis and the plane is transparent enough to see the sphere. Note that this sphere is not admissible, as defined above, showing that Theorem 1 can be generalized further.

In our own experience, computer visualization of mathematical concepts is an insightful tool for both research and education. The reaction to *Möbius Transformations Revealed* demonstrates the breadth of its appeal.

References

[1] H. Gray Funkhouser, A note on a tenth century graph, *Osiris* **1** (1936), 260–262.

[2] Tristan Needham, *Visual Complex Analysis*, The Clarendon Press, Oxford University Press, New York, 1997.

[3] Frank Farris, review of *Visual Complex Analysis*, *American Mathematical Monthly* **105** (1998), 570–576.