Stability, Consistency, and Convergence:
A 21st Century Viewpoint

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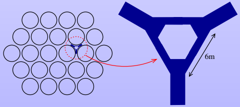
冯康 1920–1993

Feng's significance for the scientific development of China cannot be exaggerated. He not only put China on the map of applied and computational mathematics, through his own research and that of his students, but he also saw to it that the needed resources were made available. . . .
Many remember his small figure at international conferences, his eyes and mobile face radiating energy and intelligence. He will be greatly missed by the mathematical sciences and by his numerous friends.

– Peter Lax, writing in SIAM News, 1993

The failure of the Sleipner A offshore platform



\$700,000,000 Richter magnitude 3

Convergence, consistency, and stability of discretizations

$L : X \to Y$ bounded linear operator on Banach spaces.
Continuous problem: Given $f \in Y$ find $u \in X$ such that $Lu = f$.

Assume it is well-posed: $\forall f\ \exists!\ u$ s.t. $Lu = f$, $f \mapsto u$ is continuous

Discrete problem: $L_h : X_h \to Y_h$ operator on finite dimensional spaces, $f_h \in Y_h$. Find $u_h \in X_h$ such that $L_h u_h = f_h$.

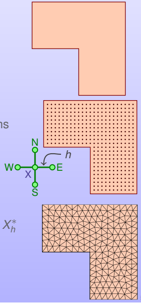
- The discretization is *convergent* if u_h is sufficiently near u.
- The discretization is *consistent* if L_h and f_h are sufficiently near L and f.
- The discretization is *stable* if the discrete problem is well-posed.

"Fundamental metatheorem of numerical analysis"

A discretization which is consistent and stable is convergent.

A simple example: Dirichlet problem for Poisson's equation

Continuous problem:
Given $f \in L^2(\Omega)$ find $u \in \mathring{H}^1_0(\Omega)$ such that
$Lu := -\Delta u = f$ in Ω.



Finite difference discretization: $X_h = Y_h =$ grid fns

$$L_h u(X) := \frac{4u(X) - u(N) - u(S) - u(E) - u(W)}{h^2}$$

$f_h = f|_{\text{grid pts}}$ $\qquad -\Delta_h u(X)$

Finite element discretization: $X_h \subset \mathring{H}^1_0(\Omega)$, $Y_h = X_h^*$

$$\langle L_h u, v\rangle_{X_h^* \times X_h} = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in X_h$$

$$\langle f_h, v\rangle_{X_h^* \times X_h} = \int_\Omega f \, v \, dx$$

Measuring convergence, consistency, and stability

To quantify convergence we use

1. A restriction operator $r_h : X \to X_h$.
2. A norm in the space X_h.

The *discretization error* is then $\|r_h u - u_h\|_{X_h}$. The method is convergent if it tends to 0 as $h \to 0$.

To quantify consistency we use a norm in the space Y_h. The *consistency error* is then $\|L_h r_h u - f_h\|_{Y_h}$. The method is consistent if it tends to 0.

The *stability constant* is $\|L_h^{-1}\|_{\mathcal{L}(Y_h, X_h)}$. The method is stable if it remains bounded as $h \to 0$.

In this context the fundamental metatheorem is a theorem:

$$L_h u_h = f_h \implies L_h r_h u - L_h u_h = L_h r_h u - f_h \implies r_h u - u_h = L_h^{-1}(L_h r_h u - f_h)$$

$$\|r_h u - u_h\|_{X_h} \le \|L_h^{-1}\|_{\mathcal{L}(Y_h, X_h)} \|L_h r_h u - f_h\|_{Y_h}$$

An elementary example of instability

Compute $\displaystyle\int_0^1 x^{14} e^{x-1} \, dx$

$$\gamma_{n+1} = \int_0^1 x^n e^{x-1} \, dx \qquad \gamma_{n+1} = 1 - n\gamma_n \qquad \gamma_1 = 1 - e^{-1} = 0.632121\ldots$$

n	γ_n	n	γ_n	n	γ_n
1	0.632121	6	0.145480	11	1.684800
2	0.367879	7	0.127120	12	-17.5328
3	0.264242	8	0.110160	13	211.394
4	0.207274	9	0.118720	14	-2747.12
5	0.170904	10	-0.068480	15	38,460.6

$$\int_0^1 x^{14} e^{x-1} \, dx = 38{,}460.6 \qquad ???$$

Finite differences for the heat equation

Initial value/boundary value problem:

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0 \qquad x \in \Omega,\ t \in [0,T]$$

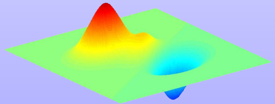
$$u(x,t) = 0 \qquad x \in \partial\Omega,\ t \in [0,T]$$

$$u(x,0) = u_0(x) \qquad x \in \Omega$$

Discretization:

$$\frac{u_h(x,t+k) - u_h(x,t)}{k} - \Delta_h u_h(x,t) = 0, \quad x \in \text{grid},\ t = 0, k, 2k, \ldots$$

consistency error = $O(k) + O(h^2)$



Stability for the discretized heat equation

$$\frac{u_h(x,(n+1)k) - u_h(x,nk)}{k} - \Delta_h u_h(x,nk) = 0$$

$$u_h(\,\cdot\,,(n+1)k) = (I + k\Delta_h)u_h(\,\cdot\,,nk) =: G[u_h(\,\cdot\,,nk)]$$

$$Gv(x) = \left(1 - \frac{4k}{h^2}\right)v(x) + \frac{k}{h^2}[v(N) + v(S) + v(E) + v(W)]$$

If $k \le h^2/4$ then $\|Gv\|_{l_\infty} \le \|v\|_{l_\infty}$, so $u_h(\,\cdot\,,0) \mapsto u_h$ is bounded in l_∞ uniformly in h and k and we obtain stability.

If $k > h^2/4$, then $\|G\|_{\mathcal{L}(l_\infty,l_\infty)} > 1$, and the method is unstable.

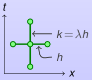
Milestones: consistency does not imply convergence

Courant, Friedrichs, and Lewy were first to realize that a consistent discretization of a well-posed problem need not converge.

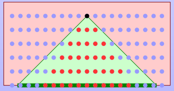
In their classic 1928 paper, they considered the wave eq.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

with centered differences and timestep k proportional to h.



If $\lambda > 1$, the numerical domain of dependence does not cover the true domain of dependence and the method cannot be convergent.



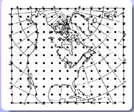
This gave us the *CFL condition*, which is necessary for convergence.

Milestones: stability

Von Neumann realized the importance of numerical stability, studying first linear algebra problems in his landmark paper with Goldstine in 1947, and then difference methods for PDEs in another landmark paper in 1950, on numerical weather prediction with Charney & Fjörtoft:

> "If the finite difference solution is to approximate closely the continuous solution, Δs and Δt must be small in comparison to the space and time scales of physically relevant motions. But this does not alone insure accuracy; the small-scale motions for which there is inevitably a large distortion may possibly be amplified in the course of the computation to such an extent that they will totally obscure the significant large-scale motions."

They went on to analyze stability for evolution equations via discrete Fourier analysis, giving birth to *von Neumann stability analysis* for a finite difference discretization of an evolution equation.



Milestones: the equivalence theorem

The convergence theory of finite difference discretizations for evolution problems was developed by many researchers in 1950s. An important capstone paper was the 1956 *Survey of the stability of linear finite difference equations* by Lax and Richtmeyer.

> "The term *stability* refers to a property of the…sequence of finite difference equations with increasingly finer mesh. We shall give a definition of stability in terms of the uniform boundedness of a certain set of operators and then show that under suitable circumstances, for linear initial value problems, stability is necessary and sufficient for convergence.… The circumstances are first…consistency…and second, that the initial value problem be properly posed."

Lax equivalence theorem

For consistent finite difference discretizations of well-posed linear initial-value problems, stability $\Longleftrightarrow$ convergence.

Finite element methods

Continuous problem:
V, W Hilbert spaces, $B : V \times W \to \mathbb{R}$ bdd bilinear form, $f \in W^*$

Find $u \in V$ such that $B(u, w) = \langle f, w \rangle$, $w \in W$.

$\iff Lu = f$ where $\langle Lv, w \rangle := B(v, w)$.

Ex: $V = W = \mathring{H}^1(\Omega)$, $B(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx$, $\langle f, v \rangle = \int_\Omega f \, v \, dx$

Discrete problem:
V_h, W_h of equal finite dimension, $B_h : V_h \times W_h \to \mathbb{R}$ bilinear, $f_h \in W_h^*$.

Find $u_h \in V_h$ such that $B_h(u_h, w) = \langle f_h, w \rangle$, $w \in W_h$.

This is a *generalized Galerkin method*. If V_h and W_h are piecewise polynomial spaces of a certain sort, it is a *finite element method*.
Ex: If $V_h \subset V$, $W_h \subset W$, $B_h = B|_{V_h \times V_h}$, $f_h = f|_{W_h}$, this is a true Galerkin method, or a *conforming FEM.*

For simplicity we henceforth assume $V = W$, $V_h = W_h \subset V$
(although both $V_h \neq W_h$ and $V_h \not\subset V$ are of interest).

Stability of finite element methods

For a generalized Galerkin method, the stability constant can be expressed as the reciprocal of the *inf-sup constant*:

$$\|L_h^{-1}\|_{\mathcal{L}(V_h^*, V_h)} = \frac{1}{\gamma_h}$$

where

$$\gamma_h := \inf_{0 \neq v \in V_h} \sup_{0 \neq w \in V_h} \frac{B_h(v, w)}{\|v\|_V \|w\|_V}.$$

So we need to bound γ_h below. This is easy if B_h is coercive:

$$B_h(v, v) \geq \gamma \|v\|^2,$$

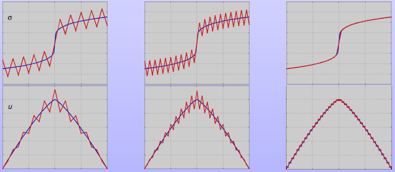
but otherwise can be quite difficult.

Mixed Laplacian in 1D

Babuška–Narasimhan

$$\sigma + u' = 0, \quad \sigma' = f \qquad \text{on } (-1, 1)$$

$$B(\sigma, u; \tau, v) := \int_{-1}^{1} (\sigma\tau - \tau' u + \sigma' v) \, dx = \int_{-1}^{1} f v \, dx \quad \forall \tau \in H^1, \; v \in L^2$$

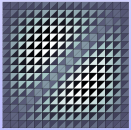


$\mathcal{P}_1$-$\mathcal{P}_1$ (20 elts) $\mathcal{P}_1$-$\mathcal{P}_1$ (40 elts) $\mathcal{P}_1$-$\mathcal{P}_0$ (40 elts)

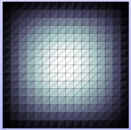
Mixed Laplacian in 2D

$$\sigma + \operatorname{grad} u = 0, \quad \operatorname{div} \sigma = f$$

$$\int_\Omega (\sigma \cdot \tau - \operatorname{div} \tau \, u + \operatorname{div} \sigma \, v) \, dx = \int_\Omega f v \, dx \quad \forall \tau \in H(\operatorname{div}), \; v \in L^2$$



$\mathcal{P}_1$–$\mathcal{P}_0$



RT_0–$\mathcal{P}_0$

Raviart-Thomas 1976

Consistency of finite element methods

For r_h take the V-orthogonal projection into V_h: $r_h u = \arg\min_{v \in V_h} \|u - v\|_V$.

$$\text{consistency error} := \|L_h r_h u - f_h\|_{V_h^*} = \sup_{w \in V_h} \frac{|B_h(r_h u, w) - \langle f_h, w\rangle|}{\|w\|_V}.$$

The simplest (and most important) case is a *conforming FEM*, i.e., $B_h = B|_{V_h \times V_h}$, $f_h = f|_{V_h}$. Then the consistency error is

$$\sup_{w \in V_h} \frac{|B(r_h u, w) - \langle f, w\rangle|}{\|w\|_V} = \sup_{w \in V_h} \frac{|B(r_h u - u, w)|}{\|w\|_V} \le \|B\| \inf_{v \in V_h} \|u - v\|_V.$$

For a conforming method the consistency error is bounded by the *approximation error* times $\|B\|$.

In the general case,

conformity error

$$\text{consistency error} \le \|B_h\| \inf_{v \in V_h} \|u - v\| + \sup_{v \in V_h} \frac{|B_h(u, w) - \langle f_h, w\rangle|}{\|w\|}$$

Convergence of finite element methods

By the fundamental theorem, for a conforming FEM we have

$$\|u_h - r_h u\|_V \le \gamma_h^{-1} \|B\| \inf_{v \in V_h} \|u - v\|_V.$$

With the triangle inequality, this becomes

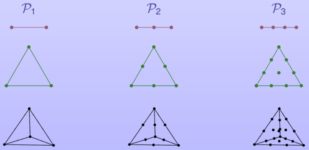
$$\|u - u_h\|_V \le (1 + \gamma_h^{-1}\|B\|) \inf_{v \in V_h} \|u - v\|_V$$

Stable conforming finite elements are quasioptimal.
(Cea 1964 in the coercive case, Babuška 1972)

For nonconforming methods we get some extra terms coming from the conformity error. (Strang 1972, coercive case)

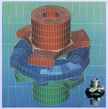
Bounds on the approximation error

Finite element spaces must be constructible from local information, the shape functions and degrees of freedom. For approximating the space H^1 using simplicial meshes, the most common finite elements are the *Lagrange* elements, consisting of all continuous functions which belong to $\mathcal{P}_r$ on each simplex.

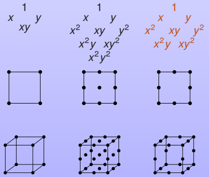


For Lagrange $\mathcal{P}_r$ elements $\inf_{v \in V_h} \|u - u\|_{H^1} = O(h^r)$
(as long as u is smooth and the simplices don't degenerate).

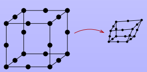
Quadrilaterals and hexahedra



On cubes, instead of $\mathcal{P}_r$ the natural choice is $\mathcal{Q}_r = \bigotimes_{i=1}^n \mathcal{P}_r(I)$. But the clever *serendipity elements* $\mathcal{P}_r \subset \mathcal{Q}_r' \subset \mathcal{Q}_r$, also give $O(h^r)$ approximation.

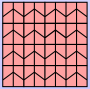


For distorted quads or hexes, we use a multilinear map from a reference cube, and compose to get the shape functions.



Reduced convergence for distorted serendipity elements

Folklore had it that $\mathcal{Q}_r$ and $\mathcal{Q}'_r$ both afford $O(h^r)$ approximation. It turns out (A-Boffi-Falk 2002), this is true on parallelpids, but only $\mathcal{Q}_r$ achieves $O(h^r)$ on general hexahedra. For $\mathcal{Q}'_r$ on general hexahedra, the convergence rate is reduced from r to $\lfloor r/2 \rfloor$.



$\mathcal{Q}_2$

n	% err.	rate
2	48.58	
4	12.08	2.0
8	3.02	2.0
16	0.75	2.0
32	0.19	2.0
64	0.05	2.0

$\mathcal{Q}'_2$

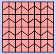
n	% err.	rate
2	51.21	
4	14.72	1.8
8	4.84	1.6
16	1.89	1.4
32	0.84	1.2
64	0.40	1.1

From *The Sleipner accident and its causes,* B Jakobsen, 1998:
"The reasons for the...reduced load bearing capacity were:

(a) Unfavorable geometrical shaping of some finite elements in the global analysis. In conjunction with the subsequent post-processing of the analysis results, this led to underestimation of the shear forces at the wall supports by some 45%.
(b) Inadequate design of the haunches at the cell joints, which support the tricell walls."

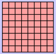
Reduced convergence for vectorial elements

In 2005 A-B-F showed the problem is even worse for vectorial elements. The popular lowest order Raviart–Thomas elements go from $O(h)$ to $O(1)$ on non-parallel quadrilaterals and hexahedra. Bermudez et al. showed a striking consequence in this computation of the fundamental eigenvalue of a square acoustic cavity:



n	λ_h	rate
8	10.3474	
16	10.2829	2.0
32	10.2670	2.0
64	10.2629	2.1

extrapolated: 10.2616



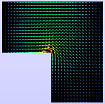
n	λ_h	rate
8	9.99708	
16	9.90136	2.0
32	9.87754	2.0
64	9.87159	2.0

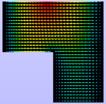
exact: $\pi^2 = 9.86906$

Vector Laplacian

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \quad \text{in } \Omega$$
$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0 \quad \text{on } \partial\Omega$$

$$\int_\Omega (\operatorname{curl} u \cdot \operatorname{curl} v + \operatorname{div} u \operatorname{div} v)\, dx = \int_\Omega f \cdot v\, dx \quad \forall v$$

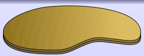




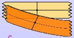
Standard finite elts converge... *but not to the soln!*
Why? The method is clearly stable in the space $H(\operatorname{curl}) \cap H(\operatorname{div})$. It is conforming, and std elts give good approximation in H^1... but not in $H(\operatorname{curl}) \cap H(\operatorname{div})$ (which contains H^1 as a closed subspace). So this method is inconsistent!

How can we solve this problem? Introduce $\sigma = \operatorname{curl} u$ as a new variable, and solve the resulting system using appropriate elements for $H(\operatorname{curl})$ and $H(\operatorname{div})$.

Thin elastic plates

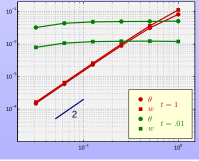


An elastic plate is a flat structural element whose thickness t is much less than the dimensions of its midsurface $\Omega \subset \mathbb{R}^2$.
Reissner–Mindlin plate equations (simplified):
Find $(\theta, w) \in H^1_0(\Omega, \mathbb{R}^2) \times H^1_0(\Omega)$ s.t.



$$\int_\Omega \nabla_S\, \theta \cdot \nabla_S\, \psi + t^{-2}(\nabla w - \theta)\cdot(\nabla v - \psi)\, dx = \int_\Omega f v\, dx \quad \forall(\psi, v)$$

It was found that they were very difficult to simulate, because standard finite elements lead to large errors when t is small. This was phemenon was named locking.



The causes and cure of locking

$$B(\theta, w; \psi, v) = \int_\Omega \nabla_S \theta \cdot \nabla_S \psi + t^{-2}(\nabla w - \theta) \cdot (\nabla v - \psi)\, dx$$

Where does the error come from? The bilinear form is H^1 coercive (hence stability), the method is conforming and linear elements give good approximation in H^1... but $\|B\|$ behaves like t^{-2}, so we still have a large consistency error.

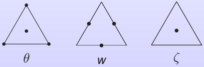
How can we solve the problem? Introduce $\zeta = t^{-2}(\nabla w - \theta)$, the shear stress. The new bilinear form is

$$B(\theta, w, \zeta; \psi, v, \eta) = \int_\Omega \nabla_S \theta \cdot \nabla_S \psi + \zeta \cdot (\nabla v - \psi) - (\nabla w - \theta) \cdot \eta + t^2 \zeta \cdot \eta\, dx$$

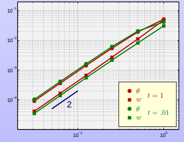
which doesn't blow-up as $t \to 0$. The difficulty is that this form is not coercive, and it is tricky to find stable elements.

The simplest locking free plate elements

The AF element (1989):



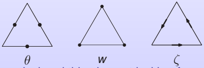
We introduced the and additional DOF for θ and nonconformity for w exactly in order to achieve stability. We have to bound the nonconformity error, but it turns out that this is easy to estimate, and so we proved that *the AF element converges as $h \to 0$, uniformly in t.*



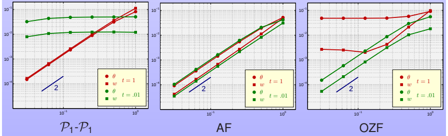
Additional locking-free elements have since been devised, although none quite as simple.

An attempt at another simple locking free elt

Oñate-Zarate-Flores (1994):

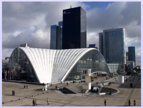


On the basis of experiments they declared this element locking-free. Again the nonconformity error is controllable. However stability is subtle. A careful analysis (A–Falk 1997) showed that the inf-sup constant behaves like $\min(1, h^2/t^2)$. Thus we have stability if $t = O(h)$, but lose stability and convergence as $h \to 0$ with t fixed.



Thin elastic shells

An elastic shell is a *curved* structural element whose thickness is much less than the dimensions of its midsurface. Shells are the primadonna of structures: they can deliver remarkable performance, but occasionally experience catastrophic collapse.



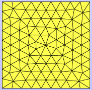
The locking problems encountered for plates are much greater for shells, and despite tremendous efforts *no one has succeeded to develop a numerical method for shells which is certifiably convergent over a wide range of conditions.* This stands as a major challenge in computational engineering.

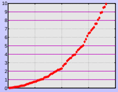
The impact of instability on an eigenvalue problem

For many electromagnetic problems, stable finite elements are elusive. A striking example of what can go wrong with unstable elements is the curl-curl eigenvalue problem: $\int \operatorname{curl} u \cdot \operatorname{curl} v \, dx = \lambda \int u \, v \, dx \ \forall v$, solved using standard $\mathcal{P}_1$ finite elements.

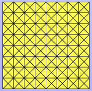
On a square the positive eigenvalues are 1, 1, 2, 4, 4, 5, 5, 8, ...

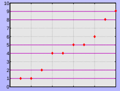
On an unstructured mesh, the discrete spectrum looks nothing like the true spectrum.





On this structured mesh the discrete eigenvalues converge to the true eigenvalues, but also to a sequence of spurious ones.





Finite element exterior calculus

"Finite element exterior calculus is an approach to the design and understanding of finite element discretizations for a wide variety of systems of partial differential equations. This approach brings to bear tools from differential geometry, algebraic topology, and homological algebra to develop discretizations which are compatible with the geometric, topological, and algebraic structures which underlie well-posedness of the PDE problem being solved."
– *Finite element exterior calculus, homological techniques, and applications,* Arnold, Falk & Winther, Acta Numerica 2006, pp. 1–155.

From the point of view of FEEC, Lagrange elements don't seem at all natural for curl curl problems. We should not discretizing a differential 1-form with elements devised for 0-forms.

The Hodge Laplacian

To get an impression of FEEC consider the Hodge Laplacian. This is the PDE most closely associated to the de Rham complex

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Hodge Laplacian: Given a k-form f find a k-form u such that

$$(dd^* + d^*d)u = f$$

The degree of well-posedness is determined by the harmonic forms $\mathfrak{H}^k := \mathfrak{Z}^k \cap (\mathfrak{B}^k)^\perp$, with dimension equal to the kth Betti number.

ker d^k range d^{k-1}

Mixed formulation of the Hodge Laplacian

Introducing $\sigma = d^*u$, we get the mixed formulation of Hodge Laplacian. If we also account for the harmonic forms, the resulting problem is always well-posed.

Continuous problem:
Given $f \in L^2\Lambda^k(\Omega)$, find $\sigma \in H\Lambda^{k-1}$, $u \in H\Lambda^k$, $p \in \mathfrak{H}^k$:

$$\begin{aligned} &\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 && \forall \tau \in H\Lambda^{k-1} \\ &\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle && \forall v \in H\Lambda^k \\ &\langle u, q \rangle = 0 && \forall q \in \mathfrak{H}^k \end{aligned}$$

The proof of well-posedness, via the inf-sup condition, relies on two fundamental results:

Hodge decomposition: $H\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}$

Poincaré inequality: $\|\omega\|_{L^2} \le c \, \|d\omega\|_{L^2}, \quad \omega \in H\Lambda^k, \ \omega \perp \mathfrak{Z}^k$

So we need to capture these in our discretization.

Discretization

To discretize we use a generalized Galerkin method based on finite dimensional subspaces $V_h^{k-1} \subset H\Lambda^{k-1}$, $V_h^k \subset H\Lambda^k$.

We require the subcomplex property $dV_h^{k-1} \subset V_h^k$ in order to obtain the

Discrete Hodge decomposition: $V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}$

where $\mathfrak{H}_h^k := (\mathfrak{B}_h^k)^\perp \cap \mathfrak{Z}_h^k$.

Generalized Galerkin method:
Find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$:

$$\begin{aligned}
&\langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle = 0 && \forall \tau \in V_h^{k-1}\\
&\langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle = \langle f, v\rangle && \forall v \in V_h^k\\
&\langle u_h, q\rangle = 0 && \forall q \in \mathfrak{H}_h^k
\end{aligned}$$

When is this discretization stable, consistent, and convergent?

Bounded cochain projections

The key property is that there must exist a bounded cochain projection.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H\Lambda^{k-1} & \xrightarrow{d^{k-1}} & H\Lambda^k & \longrightarrow \cdots \\
 & \downarrow \pi_h^{k-1} & & \downarrow \pi_h^k & \\
\cdots \longrightarrow & V_h^{k-1} & \xrightarrow{d^{k-1}} & V_h^k & \longrightarrow \cdots
\end{array}$$

- π_h^k bounded
- π_h^k a projection
- $\pi_h^k d^{k-1} = d^{k-1}\pi_h^{k-1}$

Theorem

- *If* $\|v - \pi_h^k v\| < \|v\|\ \forall v \in \mathfrak{H}^k$, *then the induced map on cohomology is an isomorphism.*
- $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \le \sup_{\substack{v\in\mathfrak{H}^k\\ \|v\|=1}} \|v - \pi_h^k v\|$
- *The discrete Poincaré inequality holds uniformly in h.*
- *The generalized Galerkin method is stable and convergent.*

Finite element differential forms: shape functions

Thus we require finite dimensional de Rham subcomplexes which admit bounded cochain projections. We will construct the V_h^k as finite element spaces assembled on simplicial meshes, and so must specify polynomial shape functions and DOFs.

A key tool is the *Koszul complex*

$$0 \longleftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \longleftarrow 0$$

with $(\kappa\omega)_x := \omega_x \lrcorner x$. It satisfies the *homotopy property*

homogeneous polys.

$$(d\kappa + \kappa d)\omega = (r+k)\omega \quad \forall \omega \in \mathcal{H}_r\Lambda^k$$

It turns out that there are precisely two "natural" families of polynomial differential forms to use as shape functions on simplices: the spaces $\mathcal{P}_r\Lambda^k$ and the spaces $\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}$

Finite element differential forms: DOFs

- On a simplex T, the space $\mathcal{P}_r\Lambda^k(T)$ has a natural set of DOFs given in terms of moments weighted by $\mathcal{P}_s^-\Lambda^l(f)$ on a faces f.
- Similarly $\mathcal{P}_r^-\Lambda^k$ has DOFs based on $\mathcal{P}_s\Lambda^l$.
- In this way we obtain the two primary families of finite element differential forms.
- The DOFs also determine cochain projections. They are not bounded, but they can be modified to become so.

Finite element differential forms/Mixed FEM

$r = 1$

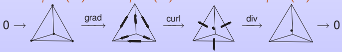
- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts
- $n = 2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Raviart–Thomas elts '76
- $n = 2$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Brezzi–Douglas–Marini elts '85
- $n = 3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 1st kind edge elts '80
- $n = 3$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd kind edge elts '86
- $n = 3$: $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 1st kind face elts '80
- $n = 3$: $\mathcal{P}_r\ \Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 2nd kind face elts '86

Finite element de Rham subcomplexes

Each of the families gives a family of finite element de Rham subcomplexes which admit bounded cochain projections.

- $\mathcal{P}_r^-\Lambda^k$ spaces of constant degree r:

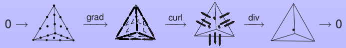
$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}) \to 0$$



Whitney 1957, Bossavit 1988

- $\mathcal{P}_r\Lambda^k$ spaces with decreasing degree:

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}) \to 0$$



Demkowicz–Monk–Vardapetyan–Rachowicz 2000

- These are extreme cases. For every r $\exists\ 2^{n-1}$ such FEdR subcomplexes.

Applications of FEEC

- Stable finite elements for the Hodge Laplacian, curl-curl problems, div-curl problems, ...
- Stable finite elements for Maxwell's equations and related EM problems
- Stable approximation of mixed eigenvalue problems
- Preconditioning and multigrid
- A posteriori error estimation

The biggest success to date is the construction of stable mixed finite elements for elasticity, which had been sought by engineers and numerical analysts for four decades. But that's another story ...