SOME CHARACTERIZATIONS OF RING EPIMORPHISMS

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It seems that at least some of these results are known by 'everyone', but they are hard to find a reference for. I will basically rewrite the proofs in Bo Stenström's *Rings and Modules of Quotients*. Rings and ring homomorphisms are always unital in this document and they form the category Ring. For a ring R, R-Mod denotes the category of left R-modules. Given a ring homomorphism φ, we denote the restriction of scalars functor by φ_* and the left adjoint of φ_* by φ^*.

We first introduce some notation before going on to these characterizations. Let $\varphi : R \to S$ be a ring homomorphism.

Let W be an S, S-bimodule. Define the *centralizer of S in W* to be

$$\mathbf{C}_S(W) = \{x \in W : sx = xs \text{ for every } s \in S\}.$$

Note that W is also an R, R-bimodule by restriction so we can also talk about the centralizer $\mathbf{C}_R(W)$. Note that in this setup we always have $\mathbf{C}_S(W) \subseteq \mathbf{C}_R(W)$. Here are some (in my opinion) clever constructions: First of all we can construct a new *ring* using S and the bimodule W:

$$\begin{bmatrix} S & W \\ 0 & S \end{bmatrix} := \left\{ \begin{bmatrix} s & x \\ 0 & t \end{bmatrix} : s, t \in S, x \in W \right\}.$$

$\begin{bmatrix} S & W \\ 0 & S \end{bmatrix}$ is a ring via matrix operations. Here the multiplication is well-defined because of the bimodule structure on W. Now given $x \in W$, consider the map

$$\begin{aligned} \beta_x : S &\to \begin{bmatrix} S & W \\ 0 & S \end{bmatrix} \\ s &\mapsto \begin{bmatrix} s & sx - xs \\ 0 & s \end{bmatrix}. \end{aligned}$$

Remark 1. $x \in \mathbf{C}_S(W)$ if and only if $\beta_x = \beta_1$. And $x \in \mathbf{C}_R(W)$ if and only if $\beta_x \circ \varphi = \beta_1 \circ \varphi$.

We claim that β_x is actually a ring homomorphism, which in turn allows us to use Remark 1 effectively in the ring epimorphism characterizations. Clearly β_x preserves the multiplicative identity. For addition, we check

$$\begin{aligned}
\beta_x(s+t) &= \begin{bmatrix} s+t & (s+t)x - x(s+t) \\ 0 & s+t \end{bmatrix} \\
&= \begin{bmatrix} s+t & sx+tx-ys-tx \\ 0 & s+t \end{bmatrix} \\
&= \begin{bmatrix} s+t & (sx-xs)+(tx-tx) \\ 0 & s+t \end{bmatrix} \\
&= \begin{bmatrix} s & sx-xs \\ 0 & s \end{bmatrix} + \begin{bmatrix} t & tx-xt \\ 0 & t \end{bmatrix} \\
&= \beta_x(s)+\beta_x(t)\,.
\end{aligned}$$

And for multiplication, we check that

$$\begin{aligned}
\beta_x(s)\beta_x(t) &= \begin{bmatrix} s & sy-ys \\ 0 & s \end{bmatrix} \begin{bmatrix} t & ty-yt \\ 0 & t \end{bmatrix} \\
&= \begin{bmatrix} st & s(ty-yt)+(sy-ys)t \\ 0 & st \end{bmatrix} \\
&= \begin{bmatrix} s & sty-yst \\ 0 & st \end{bmatrix} \\
&= \beta_x(st)\,.
\end{aligned}$$

Now we can state and prove the promised characterizations:

Theorem 2. *Let $\varphi : R \to S$ be a ring homomorphism. The following are equivalent:*

(1) *φ is an epimorphism in* Ring.
(2) *For every S,S-bimodule W, $\mathbf{C}_S(W) = \mathbf{C}_R(W)$.*
(3) *In $S \otimes_R S$, $1 \otimes s = s \otimes 1$ for every $s \in S$.*
(4) *The restriction of scalars functor φ_* is full.*
(5) *The counit $\varphi^*\varphi_* \to \mathrm{id}_{S\text{-}\mathsf{Mod}}$ is a natural isomorphism.*
(6) *The map $S \otimes_R S \to S$ given by $a \otimes b \mapsto ab$ is bijective.*

Proof. (1) $\Rightarrow$ (2): This immediately follows from Remark 1.
(2) $\Rightarrow$ (3): $S \otimes_R S$ is an S,S-bimodule via $s \cdot (a \otimes b) = sa \otimes b$ and $(a \otimes b) \cdot s = a \otimes bs$. And clearly $1 \otimes 1 \in \mathbf{C}_R(W)$ since $1 \otimes r = r \otimes 1$ for $r \in R$. Hence by assumption, $1 \otimes 1 \in \mathbf{C}_S(W)$ and we get $s \otimes 1 = s \cdot (1 \otimes 1) = (1 \otimes 1) \cdot s = 1 \otimes s$.
(3) $\Rightarrow$ (1): Let $\alpha, \beta : S \to T$ be ring homomorphisms such that $\alpha \circ \varphi = \beta \circ \varphi$. We must show that $\alpha = \beta$. Consider the map

$$\begin{aligned}
\theta : S \times S &\to T \\
(s, s') &\mapsto \alpha(s)\beta(s')\,.
\end{aligned}$$

Clearly θ is additive in both arguments. Also for $r \in R$ we have

$$\begin{aligned}\theta(s\cdot r, s') &= \theta(s\varphi(r), s')\\ &= \alpha(s\varphi(r))\beta(s')\\ &= \alpha(s)(\alpha\circ\varphi)(r)\beta(s')\\ &= \alpha(s)(\beta\circ\varphi)(r)\beta(s')\\ &= \alpha(s)\beta(\varphi(r)s')\\ &= \alpha(s)\beta(r\cdot s')\\ &= \theta(s, r\cdot s')\,.\end{aligned}$$

Thus θ induces an abelian group homomorphism

$$\begin{aligned}\lambda : S\otimes_R S &\to T\\ s\otimes s' &\mapsto \alpha(s)\beta(s')\,.\end{aligned}$$

Finally, for every $s \in S$, we get

$$\alpha(s) = \alpha(s)\beta(1) = \lambda(s\otimes 1) = \lambda(1\otimes s) = \alpha(1)\beta(s) = \beta(s)$$

as desired.
(3) $\Rightarrow$ (4): Let M and N be left S-modules and let $f : M \to N$ be an *R-linear* map. We will show that f is actually S-linear. Fix $x \in M$. Consider the map

$$\begin{aligned}\theta_x : S\times S &\to N\\ (a,b) &\mapsto af(bx)\,.\end{aligned}$$

θ_x is clearly additive in both arguments. Moreover for $r \in R$,

$$\begin{aligned}\theta_x(a\cdot r, b) &= \theta_x(a\varphi(r), b)\\ &= a\varphi(r)f(bx)\\ &= a\,(r\cdot f(bx))\\ &= af(r\cdot(bx))\\ &= af(\phi(r)bx)\\ &= af((r\cdot b)x)\\ &= \theta_x(a, r\cdot b)\,.\end{aligned}$$

Thus θ_x induces a map

$$\begin{aligned}\gamma_x : S\otimes_R S &\to N\\ (a\otimes b) &\mapsto af(bx)\end{aligned}$$

and by assumption, for every $s \in S$, we have $f(sx) = \gamma_x(1\otimes s) = \gamma_x(s\otimes 1) = sf(x)$. Thus f is S-linear.
(4) $\Rightarrow$ (5): The counit of the adjunction $\phi^* \dashv \phi_*$ at an S-module M is given by the S-linear map

$$\begin{aligned}\lambda : S\otimes_R M &\to M\\ s\otimes x &\mapsto sx\,.\end{aligned}$$

The map

$$\begin{aligned}\mu : M &\to S\otimes_R M\\ x &\mapsto 1\otimes x\end{aligned}$$

is R-linear. By assumption it must be also S-linear. We show that μ is the inverse of λ. Indeed, for every $s \in S$ and $x \in M$, we have

$$(\mu \circ \lambda)(s \otimes x) = \mu(sx) = s\mu(x) = s(1 \otimes x) = s \otimes x\,.$$

The reverse composition $\lambda \circ \mu$ is also the identity.
(5) $\Rightarrow$ (6): This is immediate by considering the natural isomorphism at the regular left module ${}_S S$.
(6) $\Rightarrow$ (3): Both $s \otimes 1$ and $1 \otimes s$ evaluate to s under the bijection, so they must be equal. $\square$