Homework # 3. MF 5012

(Due on Thursday April 10, 2008)

We consider here the application of finite difference methods to the heat equation

(P) $$\frac{\partial}{\partial t} f + \frac{\sigma^2}{2} \frac{\partial^2}{\partial S^2} f = 0 \qquad \text{in} \quad (0, S_M) \times (0, T)$$

with terminal condition

(TC) $$f(S,T) = f_T(S), \qquad \text{for } S \in (0, S_M)$$

and boundary conditions

(BC1) $$f(0,t) = f_0(t) \qquad \text{for } t \in (0,T),$$
(BC2) $$f(S_M, t) = f_{S_M}(t) \qquad \text{for } t \in (0,T).$$

For simplicity, we take $T=1$ and $S_M = 1$; we leave the volatility as a parameter we are going to define later. We are going to use uniform grids - and the notation of the notes.

(4pts) ① Consider the explicit scheme of §2 of the notes. **Write** a code to compute the approximation to the **solution of our problem (P) at time $t=0$ for any given terminal condition (TC) and any given** boundary conditions (BC1) and (BC2).

To test if the code is correct, we are going to run it in the case

$$
\begin{aligned}
f(S,T) &= -T + \frac{1}{\sigma^2} S^2 && \forall\, S \in (0, S_M) \\
f(0,t) &= -t && \forall\, t \in (0,T) \\
f(S_M,t) &= -t + \frac{1}{\sigma^2} S_M^2 && \forall\, t \in (0,T).
\end{aligned}
$$

Argue that the solution of (P) with the above boundary and terminal conditions is

$$f(S,t) = -t + \frac{1}{\sigma^2} S^2$$

Argue that the approximate solution is exact!

For what value of the ratio $\frac{\Delta t}{\Delta S^2}$ is the explicit scheme under consideration stable?

Run your code and verify that the approximate solution is indeed equal to the exact solution in the case of $\sigma^2 = 1$ and $\sigma^2 = 5$. Make sure that your choice of discretization parameters render the scheme stable.

(4pts) ② To further test if the code is correct, we consider a more complicated solution of (P), namely,

$$f(S,t) = e^{+\gamma t} \sin\left(\pi \frac{S}{S_M}\right), \quad \gamma = \frac{\sigma^2}{2} \cdot \frac{\pi^2}{S_M^2}.$$

Accordingly, we set

$$
\begin{aligned}
f(S,T) &= e^{+\gamma T} \sin\left(\pi \frac{S}{S_M}\right) && \forall\, S \in (0, S_M), \\
f(0,t) &= 0 && \forall\, t \in (0,T), \\
f(S_M,t) &= 0 && \forall\, t \in (0,T).
\end{aligned}
$$

Take $\sigma^2 = 2$ and take your discretization parameters such that $\lambda := \Delta t / \Delta S^2 = 1/4$. The purpose of this exercise is to verify that the error is of order Δt. To do this, fill the following history of convergence table:

$1/\Delta t$	$\|e^0\|$	α
16	...	—
36	...	...
64	...	...
...	...	...

The above choice of Δt corresponds to the choice $\lambda = \frac{1}{4}$ and $\Delta S = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. The parameter "$\alpha$" is an approximation to the order of the method and is obtained as follows. We assume that

$$\|e^0\|_{\Delta t} = C\, \Delta t^{\alpha}.$$

then

$$\frac{\|e^0\|_{\Delta t}}{\|e^0\|_{\Delta t'}} = \frac{\Delta t^{\alpha}}{\Delta t'^{\alpha}} = \left(\frac{\Delta t}{\Delta t'}\right)^{\alpha}$$

and so

$$\alpha \ln\left(\frac{\Delta t}{\Delta t'}\right) = \ln\left(\frac{\|e^0\|_{\Delta t}}{\|e^0\|_{\Delta t'}}\right).$$

Hence

$$\alpha = \frac{\ln\left(\frac{\|e^0\|_{\Delta t}}{\|e^0\|_{\Delta t'}}\right)}{\ln\left(\frac{\Delta t}{\Delta t'}\right)}.$$

Since we need the errors of two grids to compute one value of α, the first value of α in the table cannot be computed.

What is the value of α predicted by the theory? Are your numerical results in agreement with such prediction?

(4pts) ③ Now consider the Crank-Nicolson scheme and repeat the above exercise. Make sure to relate your numerical results to the predictions of the theory.

By comparing the tables of the history of convergence of the explicit and the Crank-Nicolson scheme, what can you say about the relative performance of those methods? Is one better than the other?

(4pts) ④ Note that the Crank-Nicolson scheme does not need to satisfy a condition of the type "($\Delta t / \Delta S^2$) is bounded". Instead of taking $\Delta t / \Delta S^2 = 1/4$, now take $\Delta t / \Delta S = C$ where "C" is some constant of your choice. Is it possible to pick "C" so that the Crank-Nicolson scheme with $\Delta t / \Delta S = C$ is more efficient than with $\Delta t / \Delta S^2 = 1/4$? Show numerical results supporting your claim.