Homework # 5 MF5012 (Due on May 1, 2008).

Consider the solution $u \in C^1(0,1)$ of the following obstacle problem:

$$\begin{aligned}
&(-u''-1)(u-f) = 0 && \text{in } (0,1),\\
&(-u''-1) \geq 0 && \text{in } (0,1),\\
&(u-f) \geq 0 && \text{in } (0,1),\\
&u(0) = u(1) = 0
\end{aligned}$$

where

$$f(x) = 1 - 8\left(x-\tfrac{1}{2}\right)^2.$$

(4pts) ① Find the exact solution of the obstacle problem.

(8pts) ② Consider the following method for numerically solving the obstacle problem

ⓐ Set $u_i^{(0)} = 2 - 8\left(i\Delta x - \tfrac{1}{2}\right)^2 \quad i = 0, \dots, I$

ⓑ Given $\{u_i^{(k)}\}_{i=0}^{I}$ compute

$$u_i^{(k+1)} = \max\left\{ f_i \,,\; u_i^{(k)} + \frac{\omega}{2}\left(\Delta x^2 + \left(u_{i+1}^{(k)} - 2u_i^{(k)} + u_{i-1}^{(k)}\right)\right)\right\}$$

for $i = 1, \dots, I-1$

$u_0^{(k+1)} = u_I^{(k+1)} = 0$. Here $f_i = f(i\Delta x)$ and $\Delta x = \frac{1}{I}$.

Show that the method converges to $\{u_i\}_{i=0}^{I}$, where

$$\begin{aligned}
&\left(-\frac{1}{\Delta x^2}\left(u_{i+1} - 2u_i + u_{i-1}\right) - 1\right)\left(u_i - f_i\right) = 0 && \text{for } i = 1, \dots, I-1,\\
&\left(-\frac{1}{\Delta x^2}\left(u_{i+1} - 2u_i + u_{i-1}\right) - 1\right) \geq 0 && \text{for } i = 1, \dots, I-1,\\
&\left(u_i - f_i\right) \geq 0 && \text{for } i = 1, \dots, I-1.
\end{aligned}$$

and

$$u_0 = u_I = 0,$$

provided $\omega \in (0,1)$.

(4pts) ③ For $I = 40$ and $\omega = \frac{1}{2}$, plot several intermediate functions $\{u_i^{(k)}\}_{i=0}^{I}$. Is the method converging? Are your results matching with the theory? Why, or why not?

(4pts) ④ Fill the table of convergence below

I	$\|e\|_I$	N	α
4			
8			
16			
32			
⋮			

where $\|e\|_I = \max_{0 \le i \le I} |u(\frac{i}{I}) - u_i|$, N is the number of iterations needed to converge and α is the estimated order of convergence. Try $\omega = \frac{1}{2}$, $\omega = 1$ and $\omega = \frac{3}{2}$. How does the variation in ω affect N and α?