Solution Guide for

HW # 2.

Problem 1.7.11

(a) A body of mass m is projected upward from the Earth's surface with an initial velocity V_0. Take the y-axis to be positive upward, with the origin on the surf. of the Earth. Assuming no air resistance and taking into account variation of the gravitational field, we obtain that

$$m \frac{dV}{dt} = -\frac{mgR^2}{(y+R)^2}$$

where R is the radius of the Earth.

Let $V(t) = v(y(t))$. Find a differential equation satisfied by $v(y)$.

Solution:

We notice that $\frac{dV}{dt} = \frac{d(v \circ y)}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt}$. Then, as we assume $m > 0$, it follows from the given equation that

$\frac{dV}{dt} = -\frac{gR^2}{(y+R)^2} \Rightarrow \frac{dv}{dy} \cdot \frac{dy}{dt} = -\frac{gR^2}{(y+R)^2}$. Also, $\frac{dy}{dt} = V(t) = v(y(t))$ by definition of V, and we have:

$$\frac{-gR^2}{(y+R)^2} = v \cdot \frac{dv}{dy}.$$

Answer: $v \cdot \frac{dv}{dy} = \frac{-gR^2}{(y+R)^2}$

(b) Find the escape velocity of the body.

Solution:

We firstly solve the diff. eq. for v as a function of y:

$$v \cdot \frac{dv}{dy} = -\frac{gR^2}{(y+R)^2} \iff$$

$$\iff \frac{d\left(\frac{v^2}{2}\right)}{dy} = -\frac{gR^2}{(y+R)^2} \iff$$

$$\iff \int_0^y \frac{d\left(\frac{v^2}{2}\right)}{dz}\, dz = -gR^2 \int_0^y \frac{1}{(z+R)^2}\, dz \iff$$

$$\iff \frac{v^2(y)}{2} - \frac{v^2(0)}{2} = -gR^2\left(-\frac{1}{y+R} + \frac{1}{R}\right) \iff$$

$$\iff \frac{v^2(y)}{2} = \frac{-ygR}{y+R} + \frac{v^2(0)}{2}.$$

Then, as $v(0) = V_0$ is our initial condition (because we assume $y=0$ ~~only~~ iff $t=0$), we get:

$$\begin{cases} \frac{v^2}{2} = -\frac{ygR}{y+R} + \frac{V_0^2}{2} \\ v(0) = V_0 \end{cases} \iff \begin{cases} v = \pm\sqrt{V_0^2 - \frac{2ygR}{y+R}} \\ v(0) = V_0 \end{cases} \iff$$

as we assume $V_0 > 0$

$$\iff v = \sqrt{V_0^2 - \frac{2ygR}{y+R}}.$$

~~In order for body to escape, it suffices to show that~~

We notice that in order for body to "escape", V should exist for all $y > 0$; that is $V_0^2 > \frac{2ygR}{y+R}$ $\forall y \in R^+$. Moreover, it is a sufficient condition, as then $V(y) > 0$ $\forall y \in R^+$. But as $\frac{y}{y+R} < 1$ and $\lim_{y \to \infty} \frac{y}{y+R} = 1$, $V_0^2 > \frac{2ygR}{y+R}$ $\forall y \in R^+$ iff $V_0^2 \geq 1 \cdot 2gR \iff$

$\iff |V_0| \geq \sqrt{2gR}$, and as we assume $V_0 > 0$, $V_0 \geq \sqrt{2gR}$.

Therefore, $V_0 = \sqrt{2gR}$ is the minimum initial velocity at which body escapes.

Answer: $V_0 = \sqrt{2gR}$

Problem 2.2.1.19

$$t^2 \frac{d^2y}{dt^2} + 2t\frac{dy}{dt} + 2y = 0, \quad t > 0.$$

Solution:

We guess the complex solution $y = t^r$, $r \neq 0$. Then we have (as $t > 0$,) so:

$$\begin{cases} t^2 r(r-1) t^{r-2} + 2trt^{r-1} + 2t^r = 0, \quad t > 0 \\ y = t^r \end{cases} \iff$$

$$\iff \begin{cases} t^r r(r-1) + t^r \cdot 2r + t^r \cdot 2 = 0, \quad t > 0 \\ y = t^r \end{cases} \iff$$

$$\iff \begin{cases} r(r-1) + 2r + 2 = 0 \\ t > 0 \\ y = t^r \end{cases} \iff \begin{cases} r^2 + r + 2 = 0 \\ t > 0 \\ y = t^r \end{cases} \iff$$

$$\Leftrightarrow \begin{cases} r = \frac{-1}{2} \pm \frac{\sqrt{7}}{2} i \\ t > 0 \\ y = t^r \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} r = \frac{-1}{2} + \frac{\sqrt{7}}{2} i \\ y = t^{-1/2} \cdot e^{(\ln t) \frac{\sqrt{7}}{2} i} \\ t > 0 \end{cases} \\ \begin{cases} r = -\frac{1}{2} - \frac{\sqrt{7}}{2} i \\ y = t^{-1/2} \cdot e^{-(\ln t) \frac{\sqrt{7}}{2} i} \\ t > 0 \end{cases} \end{array} \right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} \begin{cases} y = t^{-1/2} \cdot \left(\cos\left(\frac{\sqrt{7}}{2} \ln t\right) + i \sin\left(\frac{\sqrt{7}}{2} \ln t\right) \right) \\ r = \frac{-1}{2} + \frac{\sqrt{7}}{2} i \\ t > 0 \end{cases} \\ \begin{cases} y = t^{-1/2} \cdot \left(\cos\left(\frac{\sqrt{7}}{2} \ln t\right) - i \sin\left(\frac{\sqrt{7}}{2} \ln t\right) \right) \\ r = -\frac{1}{2} - \frac{\sqrt{7}}{2} i \\ t > 0 \end{cases} \end{array} \right.$$

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and as the initial diff. eq. is linear, the half of the sum of the roots and the half of the difference divided by i are both solutions :

$$y_1 = t^{-1/2} \cos\left(\frac{\sqrt{7}}{2} \ln t\right) \quad (t > 0)$$

$$y_2 = t^{-1/2} \sin\left(\frac{\sqrt{7}}{2} \ln t\right) \quad (t > 0).$$

These solutions are lin. independent as $W[y_1, y_2] \neq 0$ at all $t > 0$

$\Downarrow$ by exist.-uniq. thm

$y = t^{-1/2}\left(C_1 \cos\left(\frac{\sqrt{7}}{2} \ln t\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}\right) \ln t\right)$ is a ~~sol~~ general solution for the initial dif. equation,

<u>Answer:</u> $y = t^{-1/2}\left(C_1 \cos\left(\frac{\sqrt{7}}{2} \ln t\right) + C_2 \sin\left(\frac{\sqrt{7}}{2} \ln t\right)\right)$,

$C_1, C_2 \in \mathbb{R}$.

Problem 2.2.2.11

$$\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + (4t^2-2)y = 0 \qquad (*)$$

Solution:

→ We check that $y_1(t) = e^{t^2}$ is a solution to $(*)$

→ Then let $v(t) = \frac{y(t)}{e^{t^2}} \iff y(t) = v(t)e^{t^2}$. It implies

$\frac{dy}{dt} = \left(2t\,v(t) + \frac{dv}{dt}\right)e^{t^2}$ and $\frac{d^2y}{dt^2} = 2t\left(2t\,v(t) + \frac{dv}{dt}\right)e^{t^2} +$

$+ \left(2v(t) + 2t\frac{dv}{dt} + \frac{d^2v}{dt^2}\right)e^{t^2} = \left((4t^2+2)v(t) + 4t\frac{dv}{dt} + \frac{d^2v}{dt^2}\right)e^{t^2}$

$$\Downarrow$$

$\overset{0}{\overset{\shortparallel}{L(y)}} = \left(\left(\cancel{(4t^2+2)v(t)} + \cancel{4t\frac{dv}{dt}} + \frac{d^2v}{dt^2}\right) - \left(\cancel{8t^2v(t)} + \cancel{4t\frac{dv}{dt}}\right) + \cancel{4t^2v(t)} - \cancel{2v(t)}\right)e^{t^2} =$

$= \frac{d^2v}{dt^2}\cdot e^{t^2}$. But, as $e^{t^2} > 0$, it is equivalent to:

$\frac{d^2v}{dt^2} = 0 \implies v(t) = t$ is a solution.

→ But then $y_2(t) = v(t)\cdot e^{t^2} = te^{t^2}$ is a solution to $(*)$.

It is easy to see that y_1 and y_2 are lin. independent as $W[y_1, y_2]$ is non-zero on $\mathbb{R}$ $\implies$ by the exist.-uniq. Thm,

$$y(t) = C_1e^{t^2} + C_2te^{t^2}, \quad C_1, C_2 \in \mathbb{R}$$

is a general solution to $(*)$.

Answer: $y(t) = C_1 e^{t^2} + C_2 t e^{t^2}$

Problem 2.5.7

Find a particular colution to $y'' + 4y = t \sin 2t$ (*)

Solution:

→ We a method of Judicious Guessing to solve equation $y'' + 4y = te^{2ti}$ (**) As $2i$ is one of two roots of char. equation $r^2 + 4r = 0$, we guess:

$$y(t) = (at + bt^2) e^{2ti}$$

Then:
$$y'(t) = (a + 2bt + i(2at + 2bt^2)) e^{2te}$$
$$y''(t) = (2b - 4at - 4bt^2 + i(2a + 4bt + 2a + 4bt)) e^{2ti}$$

And we have:

$$(2b + i(4a + 8bt)) e^{2ti} = te^{2ti} \iff$$

$$\iff \begin{cases} 8bi = 1 \\ 2b + 4ai = 0 \end{cases} \iff \begin{cases} b = -i/8 \\ a = 1/16 \end{cases}$$

Therefore, $\dfrac{(t - 2it^2) e^{2ti}}{16}$ is a solution to (**) =>

=> its imagenary part is a solution to (*), that is:

$$y(t) = \operatorname{Im}\left(\frac{t - 2t^2 i}{16} e^{2ti}\right) = \frac{1}{16}\left(t\sin 2t - 2t^2\cos 2t\right)$$

is a particular solution to (*).

<u>Answer:</u> $y(t) = \frac{t}{16}\sin 2t - \frac{t^2}{8}\cos 2t$.

<u>Problem 2.5.15</u>

Find a particular solution to equation $\underset{L[y]}{\underbrace{y''+y}} = \cos t \cos 2t$. (*)

<u>Solution:</u>

→ We notice that $\cos t \cos 2t = \frac{\cos t}{2} + \frac{\cos 3t}{2}$ and

if $y_1(t)$ is a solution to $L_1[y] = \frac{\cos t}{2}$, and $y_2(t)$ is a solution to $L[y] = \frac{\cos 3t}{2}$, $y_1(t) + y_2(t)$ is a soln to (*) by linearity of $L[y]$.

→ For $L_1[y] = y'' + y = \frac{\cos 3t}{2}$, let us try a solution $y(t) = a\cos 3t$. Then we get ($\forall t$):

$$a(-9\cos 3t + \cos 3t) = \frac{\cos 3t}{2} \overset{\text{as } \cos 3t \neq 0 \text{ for some } t}{\Rightarrow} a = -\frac{1}{16} \text{ and}$$

$y_1(t) = \frac{-1}{16}\cos 3t$ is a solution to $L[y] = \frac{\cos 3t}{2}$.

$\rightarrow$ For $L[y] = \frac{\cos t}{2}$. We firstly consider the equation:

$y'' + y = \frac{1}{2}e^{it}$ (**) and guess a solution $y_3(t)$ with

$y_3(t) = ate^{it}$. Then we get:

$y_3''(t) = a(e^{it} + ite^{it})' = a(-t + i(1+1))e^{it}$ and

(**) $\Rightarrow ae^{it}(-t+t+i(1+1)) = \frac{1}{2}e^{it} \Leftrightarrow ai(1+1) = \frac{1}{2} \Leftrightarrow ai = \frac{1}{4} \Leftrightarrow$

$\Leftrightarrow a = \frac{-i}{4}$.

Then $y_2(t) = Re\{y_3(t)\} = \frac{1}{4}t\sin t$ is a

solution to $L[y] = \frac{\cos t}{2}$.

$\rightarrow$ Therefore, $y(t) = y_1(t) + y_2(t) = \frac{-\cos 3t}{16} + \frac{t\sin t}{4}$

is a particular soln to $(*)$.

Answer: $y(t) = -\frac{\cos 3t}{16} + \frac{t\sin t}{4}$