Solution Guide for

HW #3

Problem 2.6.1

It is found experimentally that a 1kg mass stretches a spring $\frac{49}{320}$ m. If the mass is pulled down an additional $1/4$ m and released, find the amplitude, period and frequency of the resulting motion.

Solution:

1) Let k, m be as usual, y denote the spring stretch in meters. Then we get the following equation:

$$mg - ky = my'',$$

and substituting for $y_1 = y - \frac{mg}{k}$ we get:

$$-ky_1 = my_1'' \iff my_1'' + ky_1 = 0 \overset{\text{as } k,m>0}{\iff}$$

$$\iff y_1 = a\cos\sqrt{\tfrac{k}{m}}\,t + b\sin\sqrt{\tfrac{k}{m}}\,t, \quad \text{for some } a, b\text{-const}$$

Then, by Lemma 1, this result can be written as:

$$y_1(t) = R\cos\left(\sqrt{\tfrac{k}{m}}\,t - \delta\right), \text{ where } \delta \in \left]-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right[$$

2) As $y_0 = \frac{mg}{k}$ is a stretch of the string when $ky_0 = mg$ (that is when mass m is attached), we get $y_0 = \frac{49}{320}$ and

$k = \frac{320}{49} \cdot 9.8 = 64$, $m = 1$. Moreover, as $y(0) = y_0 + 1/4$,

$y_1(0) = 1/4$, $y'(0) = y_1'(0) = 0$, and we have:

$$\begin{cases} R\cos(0-\delta) = 1/4 \\ -\sqrt{\frac{k}{m}} R \sin(0-\delta) = 0 \end{cases} \quad \text{as } R=0 \text{ isn't a sdn, } k>0 \iff \cancel{k, m > 0}$$

$$\iff \begin{cases} R\cos(-\delta) = 1/4 \\ \sin(-\delta) = 0 \end{cases} \quad \text{as } \delta \in]-\tfrac{\pi}{2}; \tfrac{\pi}{2}[\iff \begin{cases} R = 1/4 \\ \delta = 0 \end{cases}$$

and $y(t) = 1/4 \cos(8t) + \frac{49}{320}$.

3) The amplitude here is $1/4$ m, period is $\frac{2\pi}{8} = \frac{\pi}{4}$ sec

frequency is $\sqrt{\frac{k}{m}} = \sqrt{64} = 8 \ \frac{rad}{s}$

Answer: $A = 1/4$ m, $T = \frac{\pi}{4}$ sec, $f = 8 \ \frac{rad}{s}$

Problem 2.6.9

A spring-mass-dashpot system with $m=1$, $k=2$ and $c=2$ (in their respective units) hangs in equilibrium. At time $t=0$, an external force $F(t) = \pi - t$ N acts for a time interval π. Find the position of mass at anytime $t > \pi$.

Solution:

→ Let us consider the system for times $0 \le t \le \pi$.

The ~~ba~~ behavior of the system during those is given by

$$y'' + 2y' + 2y = \pi - t. \quad (*)$$

→ Using the method of Judicious Guessing we find a particular solution to $(*)$. Guessing $\underset{A,B-\text{const.}}{y = A + Bt}$, we get:

$$2Bt + (2A + 2B) = \pi - t \quad \Rightarrow$$

$$\Rightarrow \quad B = -\tfrac{1}{2}, \quad A = \frac{\pi + 1}{2}.$$

Therefore, $y_0(t) = -\frac{t}{2} + \frac{\pi+1}{2}$ is a particular soln to $(*)$.

→ Now we consider the homogeneous case of $(*)$:

$$y'' + 2y' + 2y = 0 \quad (**)$$

The char. eq is: $r^2 + 2r + 2 = 0$ has roots $-1 \pm i$ and therefore the real & imaginary parts of the complex soln $e^{(-1+i)t}$ produce two lin. independent real solns $e^{-t}\cos t$ and $e^{-t}\sin t$ to $(**)$.

(one can check lin. indep. by checking the Wronskian).

→ Therefore, the general solution to ~~$(**)$~~ $(*)$ is given by

$$y(t) = e^{-t}(a\cos t + b\sin t) - \frac{t}{2} + \frac{\pi+1}{2}.$$

Then, as the initial ~~s~~position and velocity of the system are both 0, we get $y(0)=0$ and $y'(0)=0$, and therefore:

$$\begin{cases} 0 = a + \frac{\pi+1}{2} \\ 0 = b - a - \frac{1}{2} \end{cases} \iff \begin{cases} a = -\frac{\pi+1}{2} \\ b = -\frac{\pi}{2} \end{cases},$$

as $y'(t) = e^{-t}(b\cos t - a\sin t) - e^{-t}(a\cos t + b\sin t) - \frac{1}{2} =$

$= e^{-t}((b-a)\cos t - (b+a)\sin t) - \frac{1}{2}$.

Therefore, for times $0 \le t \le \pi$ the position of the system is given by:

$$y(t) = e^{-t}\left(\frac{-\pi-1}{2}\cos t - \frac{\pi}{2}\sin t\right) - \frac{t}{2} + \frac{\pi+1}{2}.$$

→ Then, $y(\pi) = \frac{\pi+1}{2}e^{-\pi} + \frac{1}{2}$ - position at time π.

$y'(\pi) = -\frac{1}{2}(e^{-\pi}+1)$ - velocity at time π, as it must change continiously, whenever the finite forces are applied.

→ For $t \ge \pi$, the position is given by the D.E.:

$$(***)\ y'' + 2y' + y = 0, \quad y(\pi) = \frac{\pi+1}{2}e^{-\pi} + \frac{1}{2}, \quad y'(\pi) = -\frac{1}{2}(e^{-\pi}+1)$$

As shown above, $e^{-t}(a_1\cos t + b_1\sin t)$ is a generall solution to (***) and applying the initial conditions we get:

$$\begin{cases} y = e^{-t}(a_1\cos t + b_1\sin t) \\ e^{-\pi}(a_1\cdot(-1) + b_1\cdot 0) = \frac{\pi+1}{2}e^{-\pi} + \frac{1}{2} \\ (e^{-t}(a_1\cos t + b_1\sin t))'(\pi) = -\frac{1}{2}(e^{-\pi}+1) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = e^{-t}(a_1\cos t + b_1\sin t) \\ a_1 = -\left(\frac{\pi+1}{2} + \frac{e^{\pi}}{2}\right) \\ e^{-\pi}((b_1-a_1)\cos\pi - (b_1+a_1)\sin\pi) = -\frac{1}{2}(e^{-\pi}+1) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = e^{-t}(a_1\cos t + b_1\sin t) \\ a_1 = -\left(\frac{\pi+1}{2} + \frac{e^{\pi}}{2}\right) \\ a_1 - b_1 = -\frac{1}{2}(1+e^{\pi}) \end{cases} \Leftrightarrow \begin{cases} y = -e^{-t}/2\cdot((\pi+1+e^{\pi})\cos t + \pi\sin t) \\ a_1 = -\frac{1}{2}(\pi+1+e^{\pi}) \\ b_1 = -\frac{1}{2}\pi \end{cases}$$

Answer: $y(t) = \dfrac{-e^{-t}}{2}((\pi+1+e^{\pi})\cos t + \pi\sin t)$, for $t \geq \pi$,

where $y(t)$ stands for the position of the mass calculated from the point of system's equilibrium.

Problem 2.6.11

A 1kg mass is attached to a spring with spring constant $k = 4$ N/m, and hangs in equilibrium. An external force $F(t) = (1 + t + \sin 2t)$N is applied to the mass beginning at time $t = 0$. If the spring is stretched a length $(\frac{1}{2}+\frac{\pi}{4})$ m or more from its equilibrium position, then it will break. Assuming no damping present, find the time at which the spring breakes.

Solution:

→ The position of the mass related to equilibrium ~~position~~ point $y(t)$ at time t is given by:

$$y''+4y=1+t+\sin 2t \quad (*)$$

→ To find a particular solution we use the method of Judicious guessing. We guess a soln $y(t)=$ $=tAe^{2it}+Bt+C$ to the equation

$$y''+4y=e^{2it}+(1+t)i \quad (**)$$

We get: $y''(t)=A(e^{2it}+2ite^{2it})'=A(4ie^{2it}-4te^{2it})$

and

$$e^{2it}+(1+t)i=y''+4y=(A\cdot 4i)e^{2it}+4Bt+4C \Rightarrow$$

$$\Rightarrow \begin{cases} A=\frac{-i}{4} \\ B=\frac{i}{4} \\ C=\frac{i}{4} \end{cases}$$ and $y(t)=\frac{i}{4}(-te^{2it}+t+1)$ is a particular soln to $(**)$. This implies that as

$1+t+\sin 2t=\mathrm{Im}(e^{2it}+(1+t)i)$, $\mathrm{Im}(y(t))=\frac{1}{4}(-t\cos 2t+t+1)$

is a particular solution to (*).

→ Now we find a solution to homogeneous case of (*):

$$y''+4y=0 \quad (***)$$

It is obvious that e^{2it} is a complex-valued solution to (***) and therefore its real and imaginary parts, $\sin 2t$ and $\cos 2t$, are real-valued solns. As they are lin. independent (one can check Wronskian), we get:

$y_1(t) = b\sin 2t + a\cos 2t$ is a gen. soln to (***).

→ Therefore, the soln to (*) is given by:

$$y(t) = a\cos 2t + b\sin 2t + \frac{1}{4}(-t\cos 2t + t + 1)$$

Applying initial conditions ($y(0)=0$ and $y'(0)=0$), we get:

$$\begin{cases} y(t) = a\cos 2t + b\sin 2t + \frac{1}{4}(-t\cos 2t + t + 1) \\ 0 = a + \frac{1}{4} \\ 0 = 2b - \frac{1}{4} + \frac{1}{4} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y(t) = -\frac{\cos 2t}{4} + \frac{1}{4}(-t\cos 2t + t + 1) \\ a = -\frac{1}{4} \\ b = 0 \end{cases}$$

$\rightarrow$ As mass moves continiously, to find the time at which spring breaks, it is enough to find time t_0 at which $y(t_0) = \frac{1}{2} + \frac{\pi}{4}$, where t_0 is minimum such time

That is: $-\frac{\cos 2t_0}{4} + \frac{1}{4}(-t_0\cos 2t_0 + t_0 + 1) = \frac{1}{2} + \frac{\pi}{4}$

$$\Updownarrow$$

$$-(1+t_0)\cos 2t_0 + (t_0+1) = 2+\pi \iff$$

~~$\iff (1-t_0)(1-2\sin^2 t_0) + (t_0+1) = 2+\pi \iff$~~

~~$\iff -2\sin^2 t_0 (1-t_0) = \pi$~~

$$\iff (t_0+1)(1-\cos 2t_0) = 2+\pi \iff$$

$$\iff 2(t_0+1)\sin^2 t_0 = 2+\pi. \quad (4*)$$

As $\underbrace{2(t_0+1)\sin^2 t_0}_{= 2+\pi} \underset{\text{for } t_0 \geq 0}{\leq} 2(t_0+1)$, $t_0 \geq \frac{\pi}{2}$.

But for $t_0 = \frac{\pi}{2}$ we get $2(\frac{\pi}{2}+1)\sin^2\frac{\pi}{2} = 2+\pi \iff$

$\iff 0 = 0 \quad \Rightarrow \quad t_0 = \frac{\pi}{2}$ satisfies $(4*)$ and is minimum such positive t_0. Therefore $t_0 = \frac{\pi}{2}$ is time at which spring breakes.

Answer: $\frac{\pi}{2}$ sec