Solution Guide for

Homework #4

#2.11.5

Find the solution of the following initial-value problem:

$$y''+y = \begin{cases} \cos t & , 0 \le t \le \pi/2 \\ 0 & , \pi/2 \le t < \infty \end{cases} \quad ; \; y(0)=3,\; y'(0)=-1$$

Solution:

→ Taking the laplace transform of the lhs. we get:

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{y''\} + \mathcal{L}\{y\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) +$$

$$+\mathcal{L}\{y\} = (s^2+1)\mathcal{L}\{y\} - 3s + 1$$ - by Lemma 4 (p.229).

→ Taking the Laplace tranform of the r.h.s. we get:
(noticing that r.h.s $= \cos t\,(1 - H_{\pi/2}(t))$)

$$\mathcal{L}\{\text{r.h.s.}\} = \mathcal{L}\{\cos t - \cos t\, H_{\pi/2}(t)\} = \mathcal{L}\{\cos t\} - \mathcal{L}\{\cos t\, H_{\pi/2}(t)\} =$$

$$= \mathcal{L}\{\cos t\} - \mathcal{L}\{\sin(t+\pi/2)\, H_{\pi/2}(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{\sin(t-\pi/2)\cdot$$

$$\cdot H_{\pi/2}(t)\} = \mathcal{L}\{\cos t\} + e^{-\frac{\pi s}{2}}\mathcal{L}\{\sin t\} = \frac{s}{s^2+1} + e^{-\frac{\pi s}{2}} \cdot \frac{1}{s^2+1}$$ —

by proposition 3

$\Rightarrow$ Therefore, we have:

$$\mathcal{L}\{l.h.s.\} = \mathcal{L}\{r.h.s.\} \Leftrightarrow (s^2+1)\mathcal{L}\{y\} - 3s + 1 = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi s}{2}}}{s^2+1} \Leftrightarrow$$

as $s^2+1 > 0$

$$\Leftrightarrow \quad \mathcal{L}\{y\} = \frac{s}{(s^2+1)^2} + \frac{e^{-\pi s/2}}{(s^2+1)^2} + 3\frac{s}{s^2+1} - \frac{1}{s^2+1}, \text{ and}$$

$$\frac{s}{(s^2+1)^2} + \frac{e^{-\pi s/2}}{(s^2+1)^2} + 3\frac{s}{s^2+1} - \frac{1}{s^2+1} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) + e^{-\pi s/2}.$$

$$\cdot\left(\frac{s^2+1}{(s^2+1)^2} - \frac{s^2}{(s^2+1)^2}\right) + \mathcal{L}\{3\cos t - \sin t\} \overset{\text{by Proposition 1}}{=} \mathcal{L}\{\tfrac{t}{2}\sin t + 3\cos t - \sin t\} +$$

$$+ e^{-\pi s/2}\cdot\left(\tfrac{1}{2}\mathcal{L}\{\sin t\} + \frac{d}{ds}\left(\frac{1}{2}\cdot\frac{s}{s^2+1}\right)\right) \overset{\text{by Proposition 1}}{=} \mathcal{L}\{\tfrac{t}{2}\sin t + 3\cos t - \sin t\} +$$

$$+ e^{-\pi s/2}\mathcal{L}\left\{\frac{\sin t}{2} - t\cdot\frac{\cos t}{2}\right\} \overset{\text{by Property 3}}{=} \mathcal{L}\{\tfrac{t}{2}\sin t + 3\cos t - \sin t\} +$$

$$+ \mathcal{L}\left\{H_{\pi/2}(t)\cdot\left(\frac{\sin(t-\frac{\pi}{2})}{2} - (t-\tfrac{\pi}{2})\cdot\frac{\cos(t-\pi/2)}{2}\right)\right\} =$$

$$= \mathcal{L}\left\{\tfrac{t}{2}\sin t + 3\cos t - \sin t - \tfrac{1}{2}H_{\pi/2}(t)\left(\cos t + (t-\tfrac{\pi}{2})\sin t\right)\right\}.$$

$\Rightarrow$ Hence, $y(t) = \frac{t}{2}\sin t + 3\cos t - \sin t - \frac{1}{2}H_{\pi/2}(t)\left(\cos t + (t-\frac{\pi}{2})\sin t\right)$.

<u>Answer</u>: $y(t) = \frac{t}{2}\sin t + 3\cos t - \sin t - \frac{1}{2}H_{\pi/2}(t)\left(\cos t + (t-\frac{\pi}{2})\sin t\right)$.

#2.12.7

Solve the given initial-value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-3) \ ; \quad y(0) = 0 \ , \ y'(0) = 3 .$$

Solution:

→ Taking the Laplace transform of l.h.s we get:

$$\mathcal{L}\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} -$$

$$- 2y(0) + \mathcal{L}\{y\} = (s^2+2s+1)\mathcal{L}\{y\} - 3 = (s+1)^2\mathcal{L}\{y\} - 3$$

→ Taking the Laplace transform of r.h.s. we get:

$$\mathcal{L}\{e^{-t} + 3\delta(t-3)\} = \mathcal{L}\{e^{-t}\} + \mathcal{L}\{3\delta(t-3)\} =$$

$$= \frac{1}{s+1} + 3 \cdot e^{-3s} \ , \quad s > -1$$

→ Therefore, we have:

$$\mathcal{L}\{\text{l.h.s.}\} = \mathcal{L}\{\text{r.h.s.}\} \Leftrightarrow (s+1)^2\mathcal{L}\{y\} - 3 = \frac{1}{s+1} + 3e^{-3s} \Leftrightarrow$$

for $s > -1$

$$\Leftrightarrow \quad \mathcal{L}\{y\} = \frac{1}{(s+1)^3} + \frac{3e^{-3s}}{(s+1)^2} + \frac{3}{(s+1)^2} \ , \quad \text{where}$$

$$\frac{1}{(s+1)^3} + \frac{3e^{-3s}}{(s+1)^2} + \frac{3}{(s+1)^2} = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right) - 3e^{-3s}\frac{d}{ds}\left(\frac{1}{s+1}\right) -$$

~~for s>1~~

$$-3\frac{d}{ds}\left(\frac{1}{s+1}\right) = \mathcal{L}\{\tfrac{1}{2}t^2 e^{-t}\} + 3e^{-3s}\mathcal{L}\{te^{-t}\} + 3\mathcal{L}\{te^{-t}\} =$$

$$= \mathcal{L}\{\tfrac{1}{2}t^2 e^{-t} + 3te^{-t}\} + \mathcal{L}\{3H_3(t)\cdot(t-3)e^{3-t}\} =$$

$$= \mathcal{L}\{e^{-t}\left(\tfrac{t^2}{2} + 3t + 3H_3(t)\,(t-3)\,e^3\right)\}$$

$\Rightarrow$ Hence, $y(t) = e^{-t}\left(\frac{t^2}{2} + 3t + 3H_3(t)\,(t-3)\,e^3\right)$

<u>Answer</u>: $y(t) = e^{-t}\left\{\frac{t^2}{2} + 3t + 3H_3(t)(t-3)e^3\right)$.

#2.13.11

Use Theorem 9 to invert the Laplace transform:

$$\frac{1}{s^2(s+1)^2}.$$

<u>Solution</u>:

$$\Rightarrow \frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\cdot\frac{1}{(s+1)^2} = \frac{d}{ds}\left(-\frac{1}{s}\right)\cdot\frac{d}{ds}\left(-\frac{1}{s+1}\right) =$$

$$= \frac{d}{ds}\left(\mathcal{L}\{-1\}\right)\cdot\frac{d}{ds}\left(\mathcal{L}\{e^{-t}\}\right) = \mathcal{L}\{t\}\cdot\mathcal{L}\{te^{-t}\} =$$

$= \mathcal{L}\{t * te^{-t}\}$ - by Theorem 9.

$\Rightarrow$ $t * te^{-t} = \int_0^t (t-u)\, ue^{-u}\, du = t\int_0^t ue^{-u} du +$

$+ (u^2e^{-u})\Big|_0^t - 2\int_0^t ue^{-u} du = (t-2)\int_0^t ue^{-u} du + t^2e^{-t} =$

$= (t-2)\left((-ue^{-u})\Big|_0^t + \int_0^t e^{-u} du\right) + t^2e^{-t} = (t-2)(-te^{-t} -$

$- e^{-t} + 1) + t^2e^{-t} = te^{-t} + 2e^{-t} + t - 2 =$

$= e^{-t}(t+2) + (t-2)$.

$\Rightarrow$ Hence, $\mathcal{L}^{-1}\{\frac{1}{s^2(s+1)^2}\} = \mathcal{L}^{-1}\{\mathcal{L}\{e^{-t}(t+2) + (t-2)\}\} =$

$= e^{-t}(t+2) + (t-2)$.

Answer: $e^{-t}(t+2) + (t-2)$