Solution guide for Homework #5

3.8.3

Find all the solutious of the given differential equation:

$$\dot{x} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} x$$

Solution:

→ Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$. We firstly find all the eigenval. of A and corresponding eigenvectors:

the char. polyn. for A, $p(t) = \det(tI - A) = \det\begin{pmatrix} t-3 & -2 & -4 \\ -2 & t & -2 \\ -4 & -2 & t-3 \end{pmatrix}$

$$= 2\det\begin{pmatrix} -2 & -4 \\ -2 & t-3 \end{pmatrix} + t\det\begin{pmatrix} t-3 & -4 \\ -4 & t-3 \end{pmatrix} + 2\det\begin{pmatrix} t-3 & -2 \\ -4 & -2 \end{pmatrix} =$$

$$= 4(6-2t-8) + t((t-3)^2 - 16) = t^3 - 6t^2 - 15t - 8 =$$

$$= (t+1)(t^2-7t-8) = (t+1)^2(t-8).$$

Hence, $\lambda_1 = -1$ and $\lambda_2 = 8$ are the only eigenvalues of A.

→ It is clear that $\bar{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\bar{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are two lin, independent eigenvectors corresponding to λ_1.

For λ_2, ~~we~~ if $\bar{v}_3$ is an eigenvector, it is clear that we may take $\bar{v}_3 = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$.

$\rightarrow$ Thus, $e^{-t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $e^{8t}\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$ are 3 lin. independent solutions to $\dot{x} = Ax$ => by Exist.-Uniq. Thm

$$\underline{\underline{x(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{8t}\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}}}$$

is a general soln to the original equation.

#3.9.5

Solve the given initial-value problem:

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution:

$\rightarrow$ Let $A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$. Then the char. polynomial of A is given by: $p(t) = (t-1)(t+3) + 5 = t^2 + 2t + 2 =$

$= (t + (1+i))(t + (1-i))$.

Thus, $\lambda_1 = (-1-i)$ and $\lambda_2 = (-1+i)$ are the only eigenvalues of A.

$\rightarrow$ We find a complex eigenvector $\overline{v}_2$ corresponding to λ_2; let $\overline{v_2} = \begin{pmatrix} a+bi \\ c+di \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$. Then we have:

$$\begin{pmatrix} -2+i & 1 \\ -5 & 2+i \end{pmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff$$

$$\iff \begin{cases} (-2a-b+c) + i(a-2b+d) = 0 \\ (-5a+2c-d) + i(-5b+c+2d) = 0 \end{cases} \iff$$

$$\iff \begin{cases} b = c-2a \\ d = 2b-a \\ 0 = 0 \\ 0 = 0 \end{cases} \iff \begin{cases} c = b+2a \\ d = 2b-a \end{cases}$$

Taking $a=1$, $b=0$ we get that $\overline{v}_2 = e^{(-1+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ is a complex-valued solution and hence $Re\{\overline{v_2}\}$ and $Im\{\overline{v_2}\}$ are both real-valued solns:

$$Re\{\overline{v_2}\} = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}; \quad \cancel{Re}\ Im\{\overline{v_2}\} = e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Comparing values of these functions of t at 0 and 1 we get that they are linearly independent.

Hence, $\bar{x}(t) = C_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$ gives

a general solution to equation.

→ Applying initial data $\bar{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we get:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \bar{x}(0) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}.$$

Therefore $\bar{x}(t) = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}$ is a soln to initial-value problem.

#3.10.7

Solve the initial-value problem:

$$\dot{x} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solution:

→ Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$. Then the char. polynomial for A is given by: $p(t) = \det\begin{pmatrix} (t-1) & -2 & 3 \\ -1 & (t-1) & -2 \\ -1 & 1 & (t-4) \end{pmatrix} =$

$= (t-1)(t^2-5t+4+2) + 2(4-t-2) + 3(-1+t-1) =$

$= t^3 - 6t^2 + \overset{12}{\cancel{10}}t - 8 = (t-2)^3$

Thus, 2 is the only eigenvalue of A.

→ Let $\bar{v}_1$ be a corresponding eigenvector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then we get: $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$

$\iff \begin{cases} a - 2b + 3c = 0 \\ -a + b - 2c = 0 \\ -a + b - 2c = 0 \end{cases} \iff \begin{cases} b = c \\ c = -a \end{cases}$

Thus, all such eigenvectors have a form $k\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow$

$\Rightarrow$ there exists only one eigenvector for 2 which is lin. independent; WLOG assume $\bar{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Then $\bar{x}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is a solution.

→ Proceeding to generalized eigenspace we find $\bar{v}_2 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ lin. independent from $\bar{v}_1$, s.t.

$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}^2 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \iff \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = 0 \iff$

$\Leftrightarrow$ $b_1 = c_1$, and $\bar{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a eigenvector lin. indep. from $\bar{v}_1$, satisfying $(2I - A)^2(\bar{v}_2) = \bar{0}$.

Then, $\bar{x}_2 = e^{2t}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mp t \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} t \\ 1+t \\ 1+t \end{pmatrix}$ is another soln to $\dot{x} = Ax$.

$\Rightarrow$ Finally, we find $\bar{v}_3$ lin. indep. from $\bar{v}_1$ and $\bar{v}_2$,

s.t. $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}^3 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, where $\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ stands for $\bar{v}_3$.

We have: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, we can choose $\bar{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\bar{x}_3 = e^{2t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mp t \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t^2 \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) =$

$= e^{2t}\begin{pmatrix} t^2 \mp 3t \\ -t^2 + 2t \\ -t^2 + 2t + 1 \end{pmatrix}$ is another soln to $\dot{x} = Ax$.

$\to$ Checking for values at 0, it is obvious that $\bar{X}_1, \bar{X}_2, \bar{X}_3$ are lin. independent.

Then the general soln is given by:

$$\bar{X}(t) = e^{2t}\left(C_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -t \\ 1+t \\ 1+t \end{pmatrix} + C_3 \begin{pmatrix} t^2 \overset{-}{\cancel{+}} 3t \\ -t^2+2t \\ -t^2+2t+1 \end{pmatrix} \right).$$

$\to$ Using $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ we get:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cancel{0} = \bar{X}(0) = e^{2t}\left(C_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \text{ and}$$

hence $C_1 = C_2 = 1$ and $C_3 = 0$.

Thus, $x(t) = e^{2t} \begin{pmatrix} 1 \overset{-}{\cancel{+}} t \\ +t \\ +t \end{pmatrix}$ is a soln to original problem.