Solution Guide

For homework # 6.

4.7.1

$$\dot{x} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix} x$$

The char. polyn. of $A = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$ is given by:

$$p(t) = \det\begin{pmatrix} -5-t & 1 \\ 1 & -5-t \end{pmatrix} = t^2 + 10t + 24 = (t+6)(t+4).$$

Hence, $\lambda_1 = -6$ and $\lambda_2 = -4$ are two eigenvalues of A.

It is easy to see that $\bar{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are eigenvectors corresponding to λ_1 and λ_2 respectively.

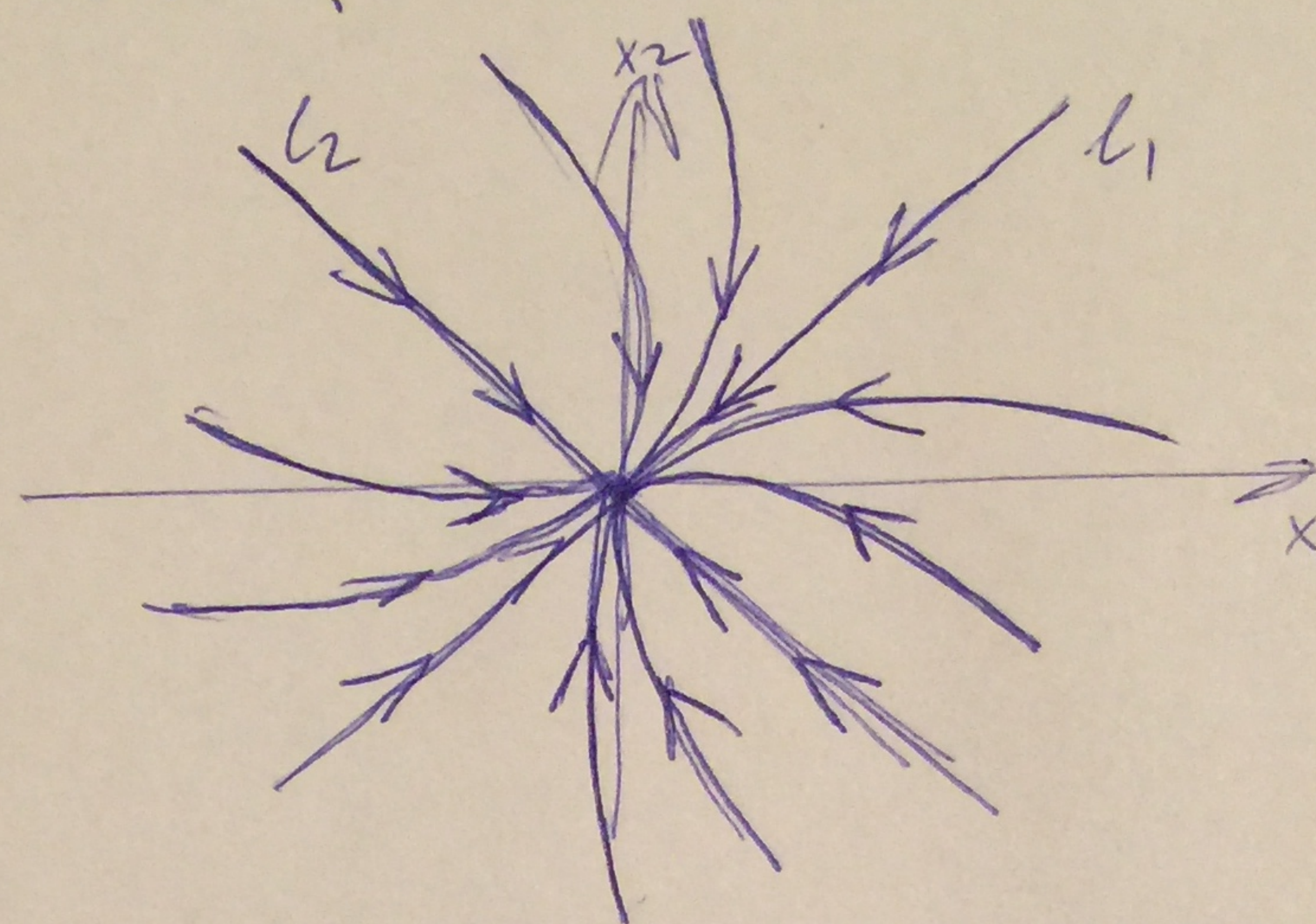
Thus, the general soln to equation is given by:

$$\bar{x}(t) = C_1 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now as both $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2 < 0$, if $l_1 = \{k\bar{v}_1 \mid k \in \mathbb{R}\} \subset \mathbb{R}^2$ and $l_2 = \{k\bar{v}_2 \mid k \in \mathbb{R}\} \subset \mathbb{R}^2$; for $t \to \infty$ and $C_1 = 0$ soln stays on l_2, for $C_2 = 0$ soln stays on l_1 and for $C_1, C_2 \neq 0$ we have

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)} = \lim_{t\to\infty} \frac{C_1 e^{-6t} + C_2 e^{-4t}}{-C_1 e^{-6t} + C_2 e^{-4t}} = \lim_{t\to\infty} \frac{C_1 e^{-2t} + C_2}{-C_1 e^{-2t} + C_2} = 1 \Rightarrow$$

$\Rightarrow$ orbit of $x(t)$ tends to align with ℓ_2, and all solns tend to 0 as $t \to \infty$:



where $(0,0)$ is an equil. point. (a stable node).

#4.7.5

$$\dot{x} = \begin{pmatrix} 1 & -4 \\ -8 & 4 \end{pmatrix} x$$

The char. polyn. is given by (for $A = \begin{pmatrix} 1 & -4 \\ -8 & 4 \end{pmatrix}$):

$$p(t) = \det \begin{pmatrix} 1-t & -4 \\ -8 & 4-t \end{pmatrix} = t^2 - 5t - 28 =$$

$$= \left(t - \frac{5+\sqrt{137}}{2}\right)\left(t - \frac{5-\sqrt{137}}{2}\right),$$

and hence the eigenvalues of A are:

$$\lambda_1 = \frac{5-\sqrt{137}}{2} \text{ and } \lambda_2 = \frac{5+\sqrt{137}}{2}, \text{ where } \lambda_1 < 0 < \lambda_2.$$

$\rightarrow$ Assuming $\bar{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ is an eigenvector corr. to λ_1

we get:

$$\begin{pmatrix} 1 - \frac{5-\sqrt{137}}{2} & -4 \\ -8 & 4 - \frac{5-\sqrt{137}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \bar{0}_{\mathbb{R}^2} \iff$$

$$\iff \begin{pmatrix} \frac{-3+\sqrt{137}}{2} & -4 \\ -8 & \frac{3+\sqrt{137}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \bar{0}_{\mathbb{R}^2} \iff$$

$$\iff \begin{pmatrix} \frac{-3+\sqrt{137}}{2} & -4 \\ \frac{-\sqrt{137}+3}{2} & \frac{128}{2\cdot\cancel{4}\cdot 4} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \bar{0}_{\mathbb{R}^2} \iff$$

$$\iff \begin{pmatrix} \frac{-3+\sqrt{137}}{2} & -4 \\ \frac{-3+\sqrt{137}}{2} & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \bar{0}_{\mathbb{R}^2}$$

And we take $\bar{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ to be $\begin{pmatrix} 8 \\ \sqrt{137}-3 \end{pmatrix}$

$\rightarrow$ Analog., if $\bar{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ corresponds to λ_2 we get:

$$\begin{pmatrix} 1 - \frac{5+\sqrt{137}}{2} & -4 \\ -8 & 4 - \frac{5+\sqrt{137}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \bar{0}_{\mathbb{R}^2} \iff \frac{-3-\sqrt{137}}{2} x_2 - 4y_2 = 0$$

and we may take $\overline{v}_2 = \begin{pmatrix} 8 \\ -\sqrt{137}-3 \end{pmatrix}$.

$\rightarrow$ Thus, the general solu is given by:

$$x(t) = C_1 e^{\frac{5-\sqrt{137}}{2}} \cdot \begin{pmatrix} 8 \\ \sqrt{137}-3 \end{pmatrix} + C_2 e^{\frac{5+\sqrt{137}}{2}} \begin{pmatrix} 8 \\ -\sqrt{137}-3 \end{pmatrix}.$$

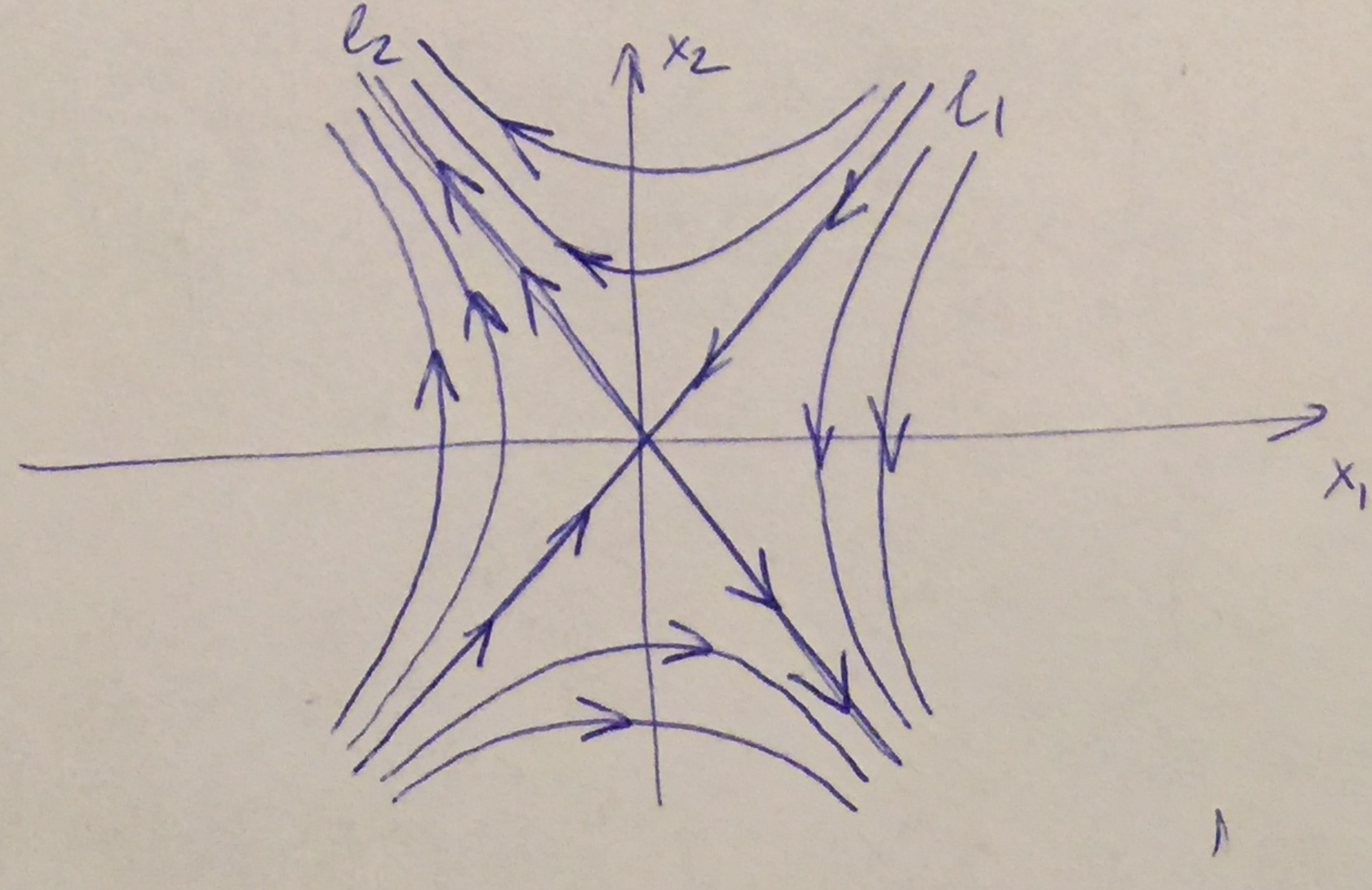
Taking $l_1 = \{k\overline{v}_1 \mid k \in \mathbb{R}\} \subset \mathbb{R}^2$ and $l_2 = \{k\overline{v}_2 \mid k \in \mathbb{R}\} \subset \mathbb{R}$

analogously to problem 1 we get:

l_1 and l_2 are orbits, for $C_1, C_2 \neq 0$ as $t \to \infty$

$x(t)$ tends closer to l_2 and as $t \to -\infty$ $|x(t)|$ tends

closer to l_1. Thus, as $\lambda_2 > 0$ and $\lambda_1 < 0$ we have:



where $(0,0)$ is a saddle point.

#4.7.9

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x$$

The char. polyn. of $A = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}$ is given by:

$$p(t) = \det \begin{pmatrix} 2-t & 1 \\ -5 & -2-t \end{pmatrix} = t^2+1 = (t+i)(t-i).$$

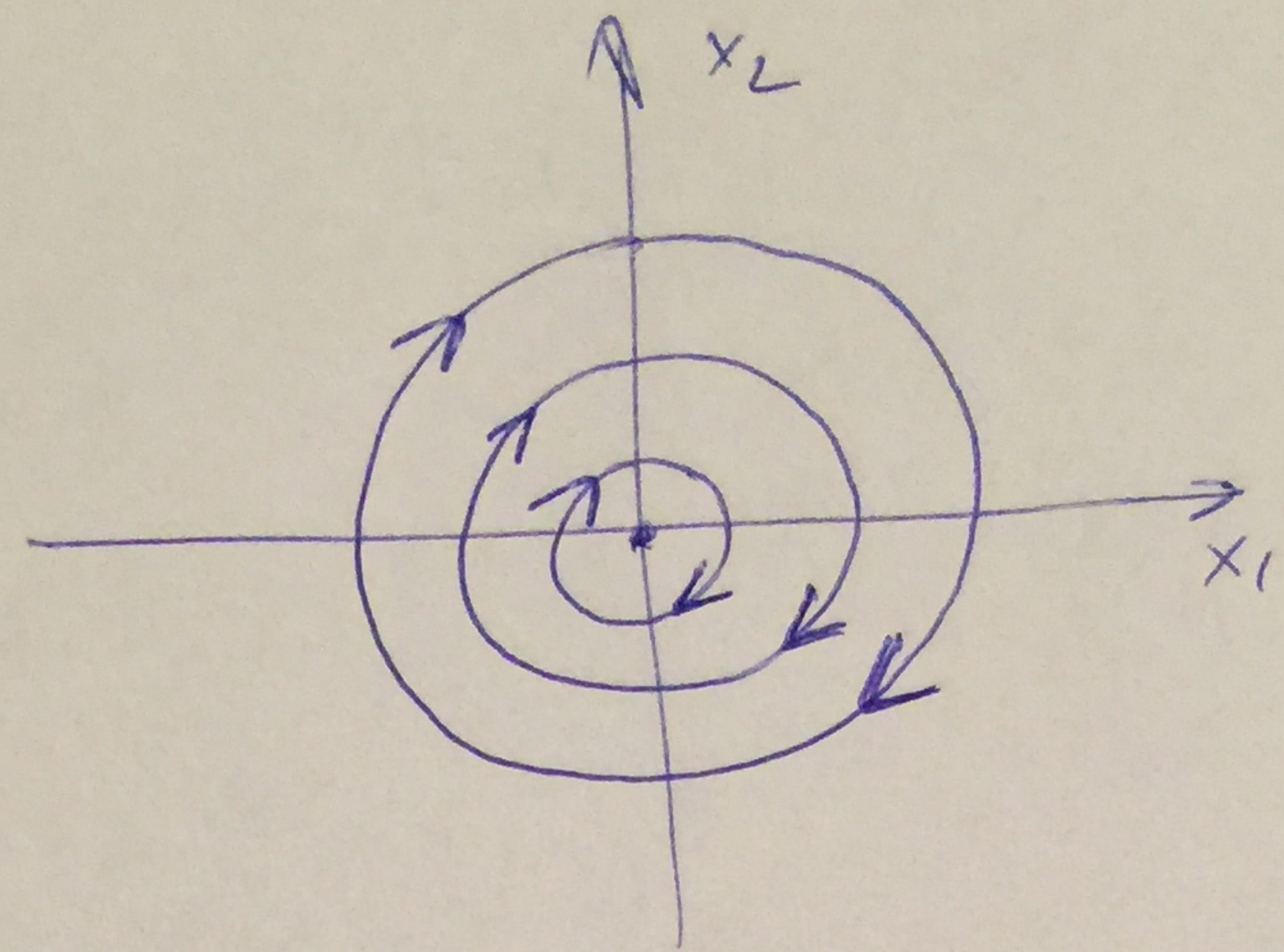
Hence $\lambda_1 = i$ and $\lambda_2 = -i$ are eigenvalues of A.

Then, by result (6) in textbook the general solu is given by:

$$x(t) = \begin{pmatrix} R_1 \cos(t-\delta_1) \\ R_2 \cos(t-\delta_2) \end{pmatrix}$$

for some $R_1 \geq 0$; $R_2 \geq 0$; $\delta_1, \delta_2 \in \mathbb{R}$.

Thus, $x(t)$ is periodic with period 2π, and $x_1(t)$ varies between $-R_1$ and R_1; $x_2(t)$ varies between $-R_2$ and R_2. Hence, the orbits are closed curves around origin (called center in this case):



where direction of orbits is clockwise as for $x_1 > 0$, $x_2 = 0$ we have (from original equation):

$$\dot{x}_2 = -5x_1 < 0 .$$